

Numerical solution for one-dimensional Richards' equation using differential quadrature method

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Nonlinear nature of Richards' equation has attracted attention for providing analytical and numerical solution for this equation. In the present study, differential quadrature method (DQM) is used for presenting a numerical solution for one-dimensional Richards' equation in mixed and h -based forms, where pressure head is the dependent variable. Our results show that DQM requires less computational effort compared to finite difference technique, and can be applied for solving other similar nonlinear equations.

Keywords: Differential quadrature method, finite difference technique, nonlinear equation, numerical solution, unsaturated zone.

AFTER an irrigation or rainfall, water flow tends to infiltrate from the soil surface into the soil due to gravity and reaches the water table. For this, water flow should infiltrate into the unsaturated zone. The flow in unsaturated zone is described by Richards' equation¹, which is a nonlinear partial differential and parabolic equation. Such an equation in one-dimensional form is described as follows:

$$h\text{-based form: } C(h) \frac{\partial h}{\partial t} = \frac{\partial}{\partial z} \left(K(h) \frac{\partial h}{\partial z} \right) - \frac{\partial K(h)}{\partial z}, \quad (1)$$

$$\Theta\text{-based form: } \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left(D(\theta) \frac{\partial \theta}{\partial z} \right) - \frac{\partial D(\theta)}{\partial z}, \quad (2)$$

$$\text{Mixed-base form: } \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left(K(h) \frac{\partial h}{\partial z} \right) - \frac{\partial K(h)}{\partial z}, \quad (3)$$

where h is the pressure head, θ the water content, $C(h)$ the specific storage coefficient, $K(h)$ the unsaturated hydraulic conductivity, $D(\theta)$ the diffusivity coefficient, z the vertical datum (positive downward) and t is the time. The nonlinear nature of Richards' equation has aroused curiosity to solve it analytically and numerically. Due to

the complexity of water flow into the unsaturated zone, and lack of a complete analytical solution to check the flow in the zone, the use of computational and numerical models to solve the problems of unsaturated flow has grown substantially in recent decades; subsurface flows are still considered to be one of the most important topics in hydrology.

Unsaturated flow modelling has been conducted by different researchers using numerical and analytical methods, each of which has its own advantages and disadvantages. Varado *et al.*² assessed an efficient numerical solution for the one-dimensional Richards' equation on bare soil. Tracy³ derived analytic solutions of Richards' equation for three-dimensional unsaturated flow for a box-shaped soil sample under simple piece-constant specified head boundary condition. Menziani *et al.*⁴ studied the analytic solution of the linearized Richards' equation for discrete arbitrary initial and boundary conditions. Shahraiyini and Ashtiani⁵ compared finite difference (FD) schemes for water flow in unsaturated soils. The aim of their study was to find an appropriate implicit numerical solution for head-based Richards' equation using some of the well-known FD schemes, including Crank Nicolson and Runge-Kutta. He and Ren⁶ examined two-dimensional solution of Richards' equation for heterogeneous soil using finite elements, and used a special method in their solution for the effective use of the FD scheme. Kuraz *et al.*⁷ worked on a solution of Richards' equation for classical and dual porosity using adaptive time discretization. Juncu *et al.*⁸ investigated two-dimensional numerical solution of Richards' equation using nonlinear multigrid method, and considering dimensionless water content as the dependent variable. They used Newton and Picard iteration methods for the solution. An exponential time-integration method was used by Carr *et al.*⁹ for solving one-dimensional Richards' equation. An iterative alternating direction implicit (IADI) algorithm for solving the equations of saturated or unsaturated flow was proposed by An *et al.*¹⁰. This algorithm is more stable than the IADI, and can be used for three-dimensional modelling. Huang and Wu¹¹ proposed analytic solutions to one-

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dimensional and vertical water infiltration in saturated or unsaturated soils that can consider the variation of rainfall with time. In their model, moisture content and permeability coefficient were assumed to be exponential functions of the pressure head, and diffusivity was considered to be constant; however, only a few analytical solutions have been proposed for Richards' equation due to its nonlinear nature. Thus, many numerical solutions have been suggested for Richards' equation by several researchers. Ginting¹² proposed a one-dimensional solution of Richards' equation using time-integration techniques. He employed the models of Haverkamp *et al.*¹³ and Van Genuchten¹⁴ as well as an exponential model. Zhu *et al.*¹⁵ developed a scheme for coupled numerical simulation of unsaturated-saturated water flow at the regional scale. In their study, saturated and unsaturated zones were implicitly coupled in space and time through the vertical flow between unsaturated soil columns and saturated aquifers in it with pressure heads in the unsaturated and saturated zones were integrated in a single matrix equation. Misiats and Lipnikov¹⁶ suggested a second-order accurate monotone finite volume scheme for Richard's equation. Paulus *et al.*¹⁷ proposed an original solution decoupling the 3D equation into 1D vertical equations and a 2D depth-integrated horizontal equation. The aim of their study was to consider vertical columns of infiltration coupled with lateral transfer of mass through the boundary conditions. Different numerical models have been proposed for modelling water flow in unsaturated-saturated zones¹⁸⁻²⁹. Here we use differential quadrature method (DQM) for solving problems of water flow in unsaturated zones. In this article, a numerical solution for nonlinear one-dimensional Richards' equation is suggested where pressure head is the dependent variable, and the mixed form of Richards' equation is investigated using DQM.

Materials and methods

Differential quadrature method

DQM is a numerical method introduced by Bellman *et al.*³⁰ for solving partial differential equations. According to them, the first-order and second-order derivatives are obtained as follows

$$f_x(x_i) = \frac{\partial f}{\partial x} \Big|_{x=x_i} = \sum_{j=1}^n a_{ij} f(x_j), \text{ For } i, j = 1, 2, \dots, n, \quad (4)$$

$$f_{xx}(x_i) = \frac{\partial^2 f}{\partial x^2} \Big|_{x=x_i} = \sum_{j=1}^n b_{ij} f(x_j), \text{ For } i, j = 1, 2, \dots, n, \quad (5)$$

where n is the number of points, x_i the coordinates of point i , and a_{ij} and b_{ij} are the first- and second-order weighting coefficients at point i for the first-order and second-order derivatives respectively. It should be noted

that the determinant of the weighting coefficients and choice of sampling points are important for the accuracy of the DQ solution. Several methods have been proposed for determining weighting coefficients. Shu³¹ described weighting coefficients for the first- and second-order derivatives as

First-order derivatives

$$a_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)} \text{ for } j \neq i, \quad (6)$$

$$a_{ii} = -\sum_{j=1}^N a_{ij} \text{ for } j = i \quad (7)$$

where $M^{(1)}(x_i)$ and $M^{(1)}(x_j)$ are defined as

$$M^{(1)}(x_i) = \prod_{j=1, j \neq i}^N (x_i - x_j), \quad (8)$$

$$M^{(1)}(x_j) = \prod_{i=1, i \neq j}^N (x_j - x_i). \quad (9)$$

Second-order derivatives

$$b_{ij} = 2a_{ij} \left(a_{ii} - \frac{1}{x_i - x_j} \right), \text{ for } j \neq i, \quad (10)$$

$$b_{ii} = -\sum_{j=1, j \neq i}^N b_{ij}. \quad (11)$$

The location of grid points has a major effect on the convergence and accuracy of the numerical results. Equally spaced grid points often yield poor results and may destroy the numerical scheme. The position of each grid point in any direction x , y and z was calculated using Chebyshev-Gauss-Lobatto method

$$Z_j = \frac{1}{2} \left(1 - \cos \left(\frac{(j-1)\pi}{N_z - 1} \right) \right) \times L_z, j = 1, 2, \dots, N_z, \quad (12)$$

where L is the length of the computational domain and N is the number of points.

Mixed-form of one-dimensional Richards' equation

The general form of this equation is based on eq. (3), which can be rewritten as

$$\frac{\partial \theta}{\partial t} = \frac{\partial K(h)}{\partial z} \frac{\partial h}{\partial z} + K(h) \frac{\partial^2 h}{\partial z^2} - \frac{\partial K(h)}{\partial z}. \quad (13)$$

Table 1. Computing $\partial K(h)/\partial z$ using DQ and FD methods

Differential quadrature method	Finite difference method
$\frac{\partial K(h)}{\partial z} = \sum_{k=1}^N a_{ik} (K(h))_k^{n+1}$	The first-order forward FD approximation for the first node
	$\frac{\partial k}{\partial z} = \frac{k_2^{n+1} - k_1^{n+1}}{Z_2 - Z_1}$
	The first-order backward FD approximation for the last node
	$\frac{\partial k}{\partial z} = \frac{k_N^{n+1} - k_{N-1}^{n+1}}{Z_N - Z_{N-1}}$
	The first-order central FD approximation for other nodes
	$\frac{\partial k}{\partial z} = \frac{k_{i+1}^{n+1} - k_{i-1}^{n+1}}{Z_{i+1} - Z_{i-1}}$

Discretizing time derivatives

$$\frac{\partial \theta}{\partial t} = \frac{\theta_i^{n+1} - \theta_i^n}{\Delta t}, \tag{14}$$

where θ_i^{n+1} and θ_i^n denote the value of θ at moments n and $n + 1$.

In the present study, discretization of the second spatial derivatives is performed by DQM, because of its advantages compared with other methods such as finite element method (FEM), FD and finite volume method. Nevertheless, discretization of time derivative is estimated using a FD approximation, as it is a common practice for other numerical solution methods (e.g. FEM, control volume method). Here also an implicit method is employed to solve the time-dependent differential equation; therefore, the solution can be considered unconditionally stable.

Discretizing $\partial K(h)/\partial z$

DQM can be used directly for discretization of $\partial K(h)/\partial z$ (Table 1), but we may come across complex functions that prevent convergence using DQM. So, in this case, we can use the FD method (Table 1).

Discretizing first- and second-order spatial derivatives

$$\left. \frac{\partial h}{\partial z} \right|_{z=z_i} = \sum_{k=1}^N a_{ik} h_k^{n+1}, \tag{15}$$

$$\left. \frac{\partial^2 h}{\partial z^2} \right|_{z=z_i} = \sum_{k=1}^N b_{ik} h_k^{n+1}. \tag{16}$$

By substituting eqs (14)–(16) as well as data presented in Table 1 into eq. (13), we get

$$\frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} = \left[\left(\frac{\partial K(h)}{\partial z} \right)_i^{n+1} \times \sum_{k=1}^N a_{ik} h_k^{n+1} + K(h) \sum_{k=1}^N b_{ik} h_k^{n+1} \right] - \left(\frac{\partial K(h)}{\partial z} \right)_i^{n+1}, \tag{17}$$

where i and K refer to positions of the points and N is the number of points in the computational domain. According to eq. (17) there are two unknowns, including θ_i^{n+1} and h_k^{n+1} . For solving this problem, Taylor series expansion of θ is used with respect to h .

$$\theta_i^{n+1,m+1} = \theta_i^{n+1,m} + \left(\frac{d\theta}{dh} \right)_i^{n+1,m} \times (h_i^{n+1,m+1} - h_i^{n+1,m}) + \text{HOT}, \tag{18}$$

where HOT represents the higher order terms, n and $n + 1$ denote the time level, while m and $m + 1$ represent the iteration level.

Computing specific storage coefficient

$d\theta/dh = C(h)$ can be computed using standard chord slope (SCS) approximation³² and tangent approximation. In tangent approximation, based on existence of an analytical relationship between θ and h for different soil types, we take the derivatives of $\theta(h)$. In the SCS method, a FD approximation is made for $d\theta/dh$, where for each trial- and error stage we have

$$C_i^n = \frac{\theta_i^{n+1} - \theta_i^n}{h_i^{n+1} - h_i^n}, \tag{19}$$

where θ_i^{n+1} and θ_i^n are the values of moisture content at times $n + 1$ and n .

Substituting eqs (18) and (19) into eq. (17), the final discretization form of eq. (3) using DQM is written as

$$\begin{aligned}
 & C(h)_i^{n+1,m} \times h_i^{n+1,m+1} \\
 & -\Delta t \left[\left(\left(\frac{\partial K(h)}{\partial z} \right)_i^{n+1,m} \times \left(\sum_{k=1}^N a_{ik} h_k^{n+1,m+1} \right) \right) \right. \\
 & \left. + K(h) \times \sum_{k=1}^N (b_{ik} h_k^{n+1,m+1}) \right] \\
 & = \left[\theta_i^{n+1} + C(h)_i^{n+1,m} \times h_i^{n+1,m} - \theta_i^n + \Delta t \left(\frac{\partial K(h)}{\partial z} \right)_i^{n+1,m} \right]. \tag{20}
 \end{aligned}$$

For discretizing $\partial K(h)/\partial z$ we use Table 1.

Head-based form of one-dimensional Richards' equation

The overall form of Richards' equation where pressure head is considered as the dependent variable is based on eq. (1), which can be rewritten as

$$C(h) \frac{\partial h}{\partial t} = \frac{\partial K(h)}{\partial z} \frac{\partial h}{\partial z} + K(h) \frac{\partial^2 h}{\partial z^2} - \frac{\partial K(h)}{\partial z}, \tag{21}$$

$$\left. \frac{\partial h}{\partial t} \right|_{z=z_i} = \frac{h_i^{n+1} - h_i^n}{\Delta t}. \tag{22}$$

If we use eqs (15)–(19), and Table 1 for discretizing the other terms of eq. (21), we get

$$\begin{aligned}
 & C(h)_i^{n+1,m} \frac{h_i^{n+1,m+1} - h_i^n}{\Delta t} \\
 & = \left[\left(\frac{\partial K(h)}{\partial z} \right)_i^{n+1,m} \times \left(\sum_{k=1}^N a_{ik} h_k^{n+1,m+1} \right) \right. \\
 & \left. + K(h) \times \sum_{k=1}^N b_{ik} h_k^{n+1,m+1} \right] - \left(\frac{\partial K(h)}{\partial z} \right)_i^{n+1,m} \tag{23}
 \end{aligned}$$

Rewriting eq. (23) gives

$$\begin{aligned}
 & C(h)_i^{n+1,m} \times h_i^{n+1,m+1} - \Delta t \left[\left(\frac{\partial K(h)}{\partial z} \right)_i^{n+1,m} \right. \\
 & \left. \times \left(\sum_{k=1}^N a_{ik} h_k^{n+1,m+1} \right) + K(h) \times \sum_{k=1}^N b_{ik} h_k^{n+1,m+1} \right] \\
 & = C(h)_i^{n+1,m} \times h_i^n - \Delta t \times \left(\frac{\partial K(h)}{\partial z} \right)_i^{n+1,m}. \tag{24}
 \end{aligned}$$

Equation (24) is the discretized form of eq. (21) using DQM, and can be solved by the algorithm presented in the next section.

Solution algorithm

First, let assume that the nonlinear system of equations in this problem is

$$[M(h)^{n+1}]_{N \times N} \{h^{n+1}\}_{N \times 1} = \{Y^{n+1}\}_{N \times 1}, \tag{25}$$

where M is a $N \times N$ matrix of coefficients and its value depends on h ; which is the matrix of unknowns containing unknown h values and a $N \times 1$ one-dimensional column matrix; Y represents a $N \times 1$ one-dimensional column matrix. Each standard numerical model such as the Newton–Raphson model can be used to solve eq. (24). One of the most important advantages of DQM compared to other methods is that we can obtain the same result using a large network with fewer points. For solving such problems, iteration-based methods can be used instead of costly computational methods like the Newton–Raphson method. In the present study, trial and error method is used

$$[M(h)^{n+1,m}]_{N \times N} \{h^{n+1,m+1}\}_{N \times 1} = \{h^{n+1,m}\}_{N \times 1}. \tag{26}$$

The hardness matrix ($[M]$) in DQM is not always a pivot matrix, as this is not the case in Galerkin FEM, finite volume and FD (pivot: the size of the member on the main diagonal position, is greater than all members in the same column and row). This fact makes the calculations a little bit more difficult; however, this difficulty can be overcome by choosing a suitable time step.

For numerically solving eq. (25), first the number of points should be selected and their location (node distribution in the computational domain) should be determined using eq. (12). By having the location of nodal points, we can compute a_{ij} and b_{ij} by using the PDQ method. Since choosing the initial guess has a major effect on the convergence of the iterative methods for solving the nonlinear system of equations, in this study, the initial condition of the problem is chosen as the initial guess (h_0). Then, $h_i^{n+1,m+1}$ value is measured using iteration according to eq. (27)

$$\{h^{n+1,m+1}\} = [M(h)^{n+1,m}]^{-1} \{h^{n+1,m}\}. \tag{27}$$

The iteration continues until the difference between two consecutive iterations is less than a specified error value (ε)

$$\text{abs}(\|h^{m+1}\| - \|h^m\|) \leq \varepsilon. \tag{28}$$

In equation (28), $\varepsilon = 0.5 \times 10^{-6}$.

Table 2. Values of the required parameters for solving examples 1 and 2

Example	θ_s	θ_r	α	β	m	n	A	k_s	γ
Example 1	0.287	0.075	1.611×10^6	3.96	–	–	1.175×10^6	0.00944	4.74
Example 2	0.368	0.102	0.335	–	0.5	2	–	0.00922	–

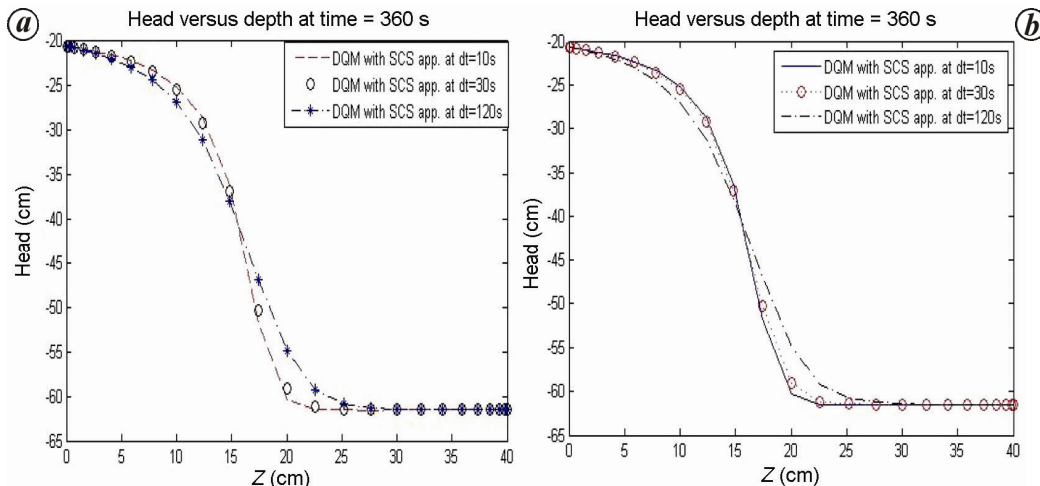


Figure 1. Numerical solution for the first example using differential quadrature method (DQM) with standard chord slope (SCS) approximation: *a*, *h*-based form; *b*, mixed form ($N = 25$, $t = 360$ s).

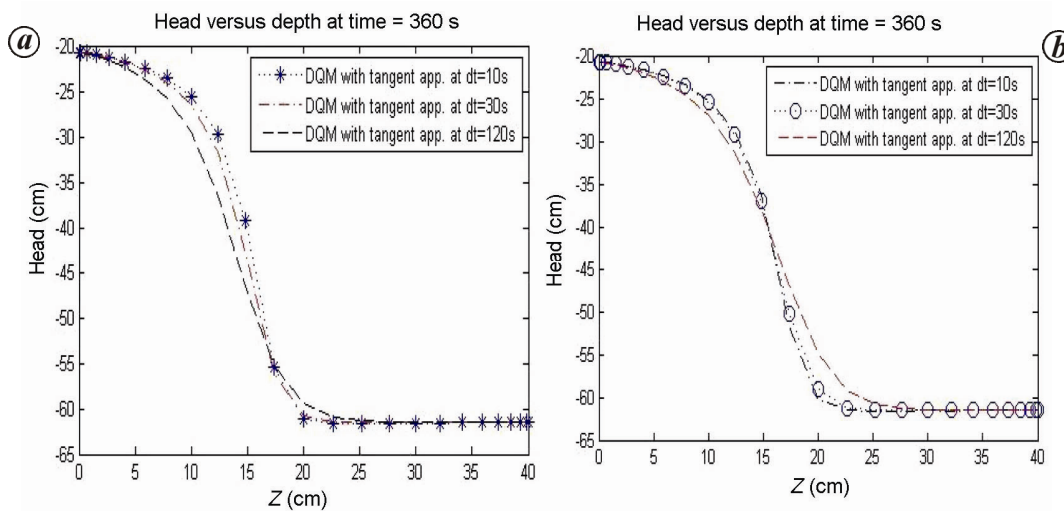


Figure 2. Numerical solution for the first example using DQM with tangent approximation: *a*, *h*-based form; *b*, mixed form ($N = 25$, $t = 360$ s)

The error in our calculations has an absolutely descending trend; therefore, a lower value of error is achievable in our calculations. However, this criterion is two orders of magnitude smaller than the value which cause the results and graphs to be independent of the error value.

Numerical examples

In this study, we solve two numerical examples which show one-dimensional water infiltration. The first example was presented by Haverkamp *et al.*¹³, where boundary

condition was -61.5 cm for downstream and -20.7 cm for upstream and initial condition was -61.5 cm. They presented eqs (29) and (30) for determining the relationship between $K(h)$ and $\theta(h)$

$$K(h) = \frac{k_s A}{A + |h|^\gamma}, \tag{29}$$

$$\theta(h) = \frac{\alpha(\theta_s - \theta_r)}{\alpha + |h|^\beta} + \theta_r. \tag{30}$$

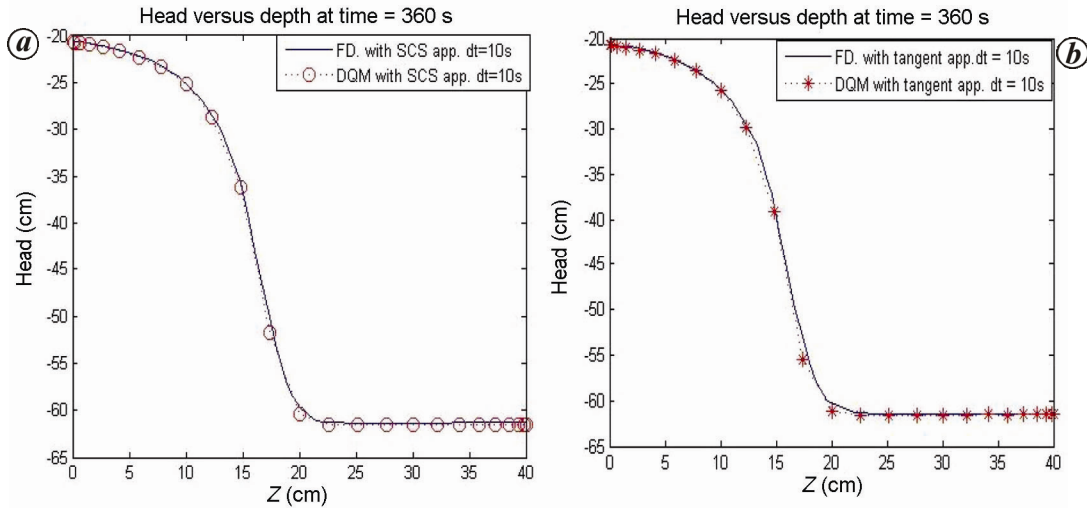


Figure 3. Comparison of finite difference (FD) and DQM results for the first example in h -based form with (a) SCS approximation and (b) tangent approximation.

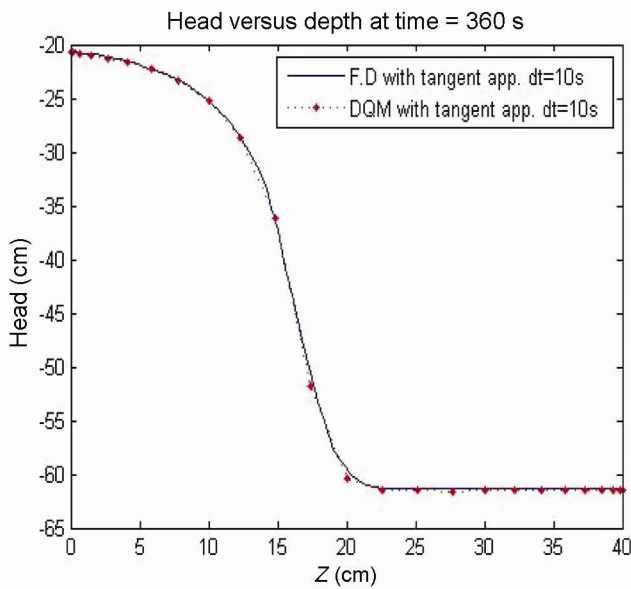


Figure 4. Comparison of FD and DQM results for the first example in mixed form with tangent approximation.

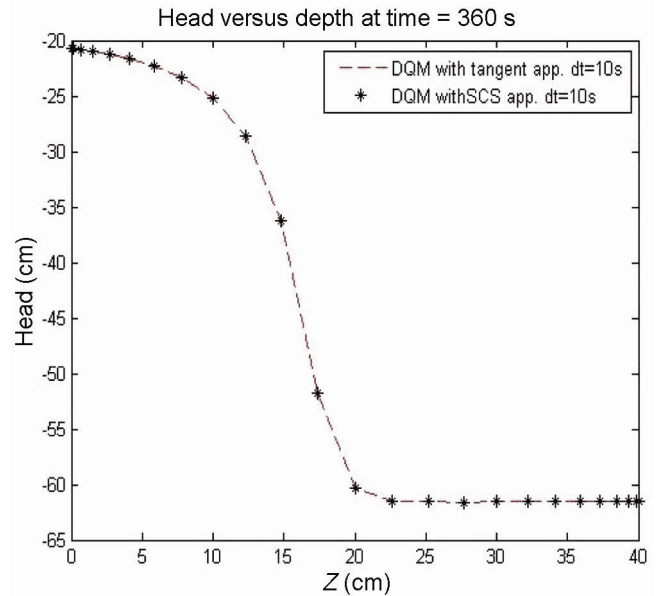


Figure 5. Comparison of SCS and tangent approximations results using DQM for solving the first example in mixed form of Richards' equation ($t = 360$ s, $N = 25$).

The second example given by Van Genuchten¹⁴ was an unsaturated, homogeneous soil column. Downstream boundary condition for this soil was -1000 cm, and upstream boundary condition was -75 cm. In addition, the initial condition was -1000 cm. Van Genuchten¹⁴ presented eqs (31) and (32) for defining the relationship between the parameters $K(h)$ and $\theta(h)$

$$K(h) = \frac{k_s \{1 - (\alpha |h|)^{n-1} [1 + (\alpha |h|)^n]^{-1+(1/n)}\}^2}{\sqrt{[1 + (\alpha |h|)^n]^{1-(1/n)}}}, \quad (31)$$

$$\theta(h) = \frac{(\theta_s - \theta_r)}{(1 + (\alpha |h|)^n)^{1-(1/n)}} + \theta_r, \quad (32)$$

where α , β , γ , m , n and A are dimensionless fitting parameters determined for different soil types, k_s denotes the saturated hydraulic conductivity, θ_s (m^3/m^3) represents moisture content of saturated soil, and θ_r (m^3/m^3) is the residual moisture content. Table 2 presents data required for this soil.

Results and discussion

First example

As mentioned above, the effective factors with regard to unsaturated zone are parameters such as hydraulic

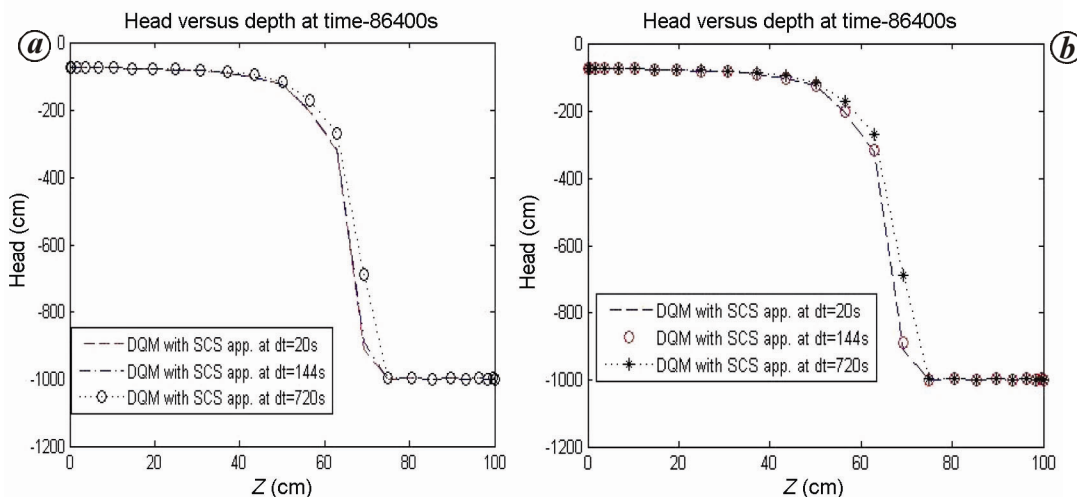


Figure 6. Numerical solution for the second example using DQM with SCS approximation: *a*, *h*-based form; *b*, mixed form ($N = 25$, $t = 360$ s).

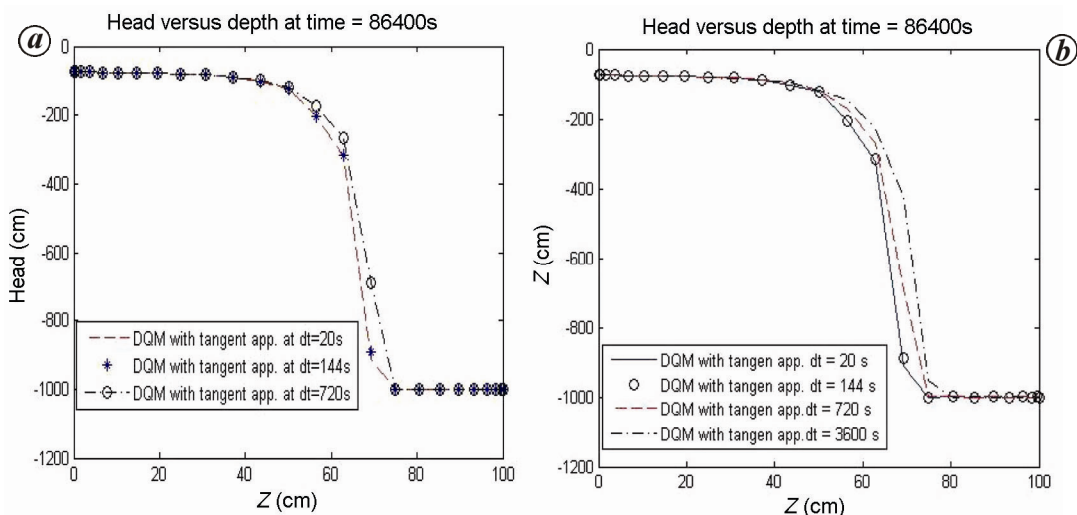


Figure 7. Numerical solution for the second example using DQM with tangent approximation: *a*, *h*-based form; *b*, mixed form ($N = 25$, $t = 360$ s).

conductivity, specific storage coefficient and moisture content. In the first example, data of which were presented by Bellman *et al.*³⁰, the problem was solved at different time steps, and SCS and tangent approximation were used for computing specific storage coefficient. The results obtained were compared with those of FD method. Also in this example, the term $\partial K(h)/\partial z$ was discretized using DQ.

Figure 1 shows results of the numerical solution for the first example in mixed and *h*-based forms of Richards' equation using SCS approximation at different time steps for $t = 360$ s; it indicates good coincident results. Figure 2 illustrates results of the first example using tangent approximation in mixed and *h*-based forms at different time steps for $t = 360$ s; the results are close

together at different time steps. Figure 3 compares DQM with fully implicit FD method for *h*-based form of Richards' equation. In Figure 3 *a*, the results of DQM and FD method are compared, where SCS approximation is used for computing specific storage coefficient. Figure 3 *b* shows a comparison of results using tangent approximation for computing specific storage coefficient. The number of points in computational domain in FD and DQM is considered as 40 and 25 respectively. The results presented in Figure 3 are in $\Delta t = 10$ s for $t = 360$ s.

Figure 4 compares the results of DQM and FD method using tangent approximation and in mixed form of Richards' equation at $\Delta t = 10$ s for $t = 360$ s. As can be seen, the results of both methods almost completely

coincide with each other. The number of points in FD and DQM is considered as 40 and 25 respectively. Figure 5 compares the results of SCS approximation with tangent approximation in mixed form of Richards' equation in $\Delta t = 10$ s for $t = 360$ s. As can be seen, the results of both methods completely coincide with each other. So, it can be concluded that the results of SCS and tangent approximation are close to each other, and both methods (DQ and FD) can be used.

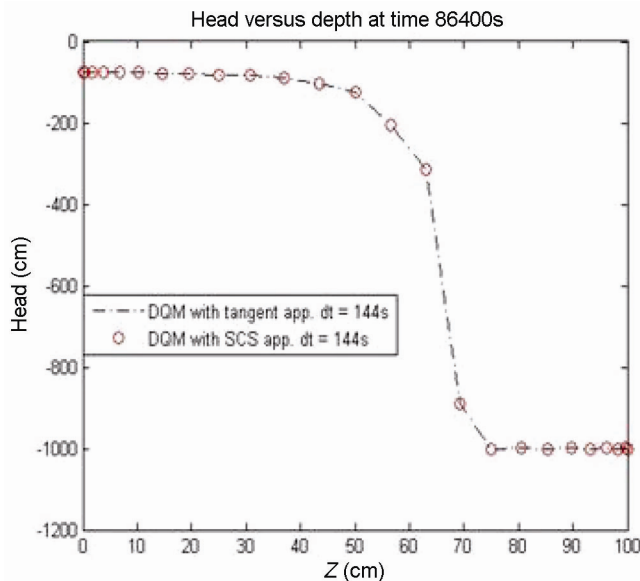


Figure 8. Comparison of SCS and tangent approximations results using DQM for solving the second example in mixed form of Richards' equation ($t = 360$ s, $N = 25$).

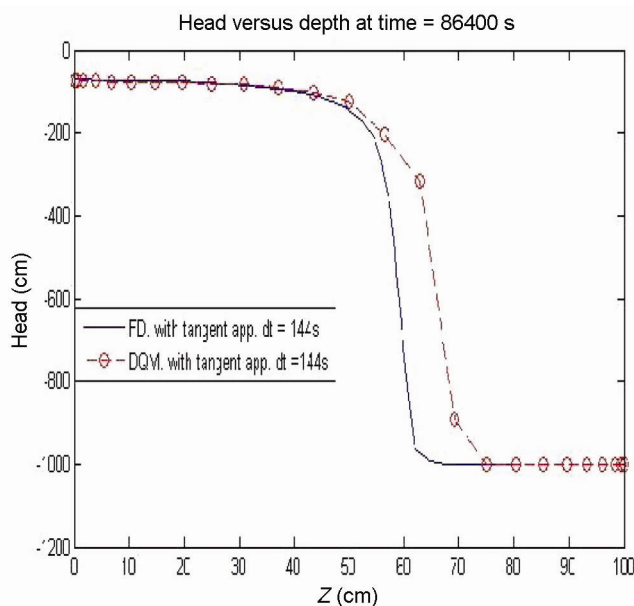


Figure 9. Comparison of FD and DQM results for the first example in mixed form with tangent approximation.

Second example

In the second example we use complex functions (eqs (31) and (32)) to determine the relationship between the parameters of $K(h)$ and $\theta(h)$ with respect to h . In this case, we may not achieve the necessary convergence for computing $\partial K(h)/\partial z$ using DQM. To solve this problem, FD method presented in Table 1 can be used for computation. Figure 6 shows results of solving the second example using DQM with SCS approximation for both h -based and mixed forms of Richards' equation at different time steps; the results obtained are coincident (different values of Delta time steps were set according to the previous studies, e.g. Krabbenhoft²⁵). Figure 7 shows DQM results with tangent approximation for both h -based and mixed forms. Since absolute error for solving the problems in this study is equivalent to 0.5×10^{-6} , the results are coincident at different time steps. Figure 8 depicts results while comparing SCS and tangent approximations for a mixed form of Richards' equation in $dt = 144$ s for $t = 86,400$ s. The results of both methods are close to each other, such that the beginning and end of the curve of both methods coincide with each other. Figure 9 shows a comparison of results using DQM and FD method for solving the second example where tangent approximation was used for computing specific storage coefficient. The numbers of points in FD and DQM are considered as 40 and 25 respectively. The results are found to coincide with each other.

Conclusion

In this study, first we discussed unsaturated zone and its importance as well as the equations of this zone known as Richards' equations. Then DQM method was introduced as an efficient method for solving differential equations, and also one which provides good results using less number of points. Then, two well-known examples given by Haverkamp *et al.*¹³ and Van Genuchten¹⁴ were presented for numerical solution of nonlinear Richards' equations. Since weighting coefficients and node distribution in computational domain are important factors for the accuracy of numerical model of DQ, selecting these two factors is of importance. Some of the results obtained by DQM were compared with those obtained by fully implicit FD method. The results confirmed the accuracy of DQM. Thus DQM is a reliable method for solving such problems and it requires less computational effort compared to the FD technique.

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