M. S. Raghunathan

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Professor M. S. Raghunathan is a world-class mathematician, and is also one of the best mathematicians India has produced, with a great international reputation. He has made foundational contributions to the theory of Lie groups, algebraic groups and their subgroups. These groups are a meeting ground for many areas of mathematics such as topology, geometry, number theory and dynamics, and Raghunathan's work has had major impact on these fields as well. The significance of his work has been recognized by the international community and he has been accorded numerous honours.

Amongst Indian mathematicians, he is only one of six Fellows of the Royal Society. He is also a *Padma Bhushan* awardee; a fellow of the World Academy of Sciences (Trieste, Italy), Indian Academy of Sciences (Bengaluru) and Indian National Science Academy (New Delhi) and is a recipient of the Shanti Swarup Bhatnagar Prize. Raghunathan was only 29 when he was invited as a speaker at the International Congress of Mathematicians (ICM) because of his significant work on discrete subgroups of Lie groups.

Background

Madabusi Santanam Raghunathan was born on 11 August 1941 into a family which prized scholarship and academic excellence. His father was a physics graduate who enjoyed occasional discussions on science with Raghunathan. He also used to regularly buy Raghunathan books (at the latter's request) on mathematics and physics. His mother was also an excellent student, but as was the custom in those days, had discontinued her studies after an early marriage.

His father became a successful wholesale timber merchant, exporting timber to Europe and Japan. Raghunathan himself had contemplated joining his father's business, but his precocity (which somehow never translated into high marks in examinations) was recognized by his teachers and relatives, who encouraged him to study sciences or the arts.

His maternal grandfather was a well-known professor of English, whose research on Thackerey was highly appreciated (his book on Thackerey was even reprinted in the US without his knowledge or permission). Perhaps this explains Raghunathan's keen interest in English literature.

While at school, Raghunathan showed keen interest in literature, mathematics, and also in running a children's magazine, as well as contributing to it. He had written a short story in Tamil, with a rather sophisticated theme for such a young author – it was about a senior scientist, who realized that a young colleague had also made the same discovery as the scientist, but did not mention this in his paper; finally he confessed to suppressing this detail on his deathbed. The story impressed the highly discerning older friends (at the Tata Institute of Fundamental Research (TIFR), Mumbai), who even urged him to publish it. He has an abiding interest in Tamil literature as also an excellent grounding in Sanskrit, not just the somewhat boring 'grammar and rules' style of learning Sanskrit, but learning classical literature and poetry. For example, he quotes extensively from famous works of Kalidasa, a large part of which he remembers to this day.

Even as a child, he used to wonder about how large numbers can be and had even discovered for himself that multiplication was commutative (e.g. that $11 \times 3 = 33$; $3 \times 11 = 33$; a fact which is not at all obvious to an 8-year-old child). This seems real evidence of mathematical precocity.

It was not all studies, though. He was also fond of cricket, and did not mind playing the game for long hours even in the summer heat of Chennai. He is also interested in painting, and is himself a capable 'doodler'.

He was too young to join the 'Intermediate' programme at Chennai and therefore studied for two years at St Joseph's College in Bengaluru. Later he enrolled as an Honours student for a B Sc degree in mathematics in Chennai. At that time he was much more interested in theoretical physics. After obtaining his B Sc degree, he was selected for the position of a research scholar at TIFR in 1960. He was one of only two candidates selected that year, among 300 applicants.

Tata Institute of Fundamental Research

When he joined TIFR, Raghunathan was very young (not yet 19) and had relatively little background in mathematics. Although he was intimidated by the scholarship of some of his young teachers, Raghunathan seemed to thrive in the relatively pressure-free atmosphere, since he became an avid and active

Raghunathan when he became a fellow of the Royal Society of London.

participant in seminars and lectures, often coming up with characteristically elegant and simple proofs which surprised even the highly talented and gifted older students/teachers. He feels deep gratitude to many of his older colleagues, often talking about the intellectual debt he owes to people at TIFR: Raghavan Narasimhan, S. Ramanan, C. P. Ramanujam, C. S. Seshadri, and M. S. Narasimhan to mention a few.

His ambition was to be a topologist. During his student days, there were many famous advances in differential and algebraic topology and he had kept abreast of the latest developments in the field. This stood him in good stead in his own work on discrete groups, although not directly. He had learnt the proof of the Atiyah–Singer index theorem, perhaps one of the deepest results of 20th century mathematics; the proof is highly intricate and involves learning a lot of mathematics in diverse fields. Raghunathan even simplified parts of the original proof using Morse theoretic methods.

His Ph D advisor, M. S. Narasimhan, himself a famous researcher with many deep and fundamental contributions to geometry, had suggested to Raghunathan the problem of deformation of connections on a Riemannian manifold. Somewhat modestly, Raghunathan did not tell him of the progress he had made on this problem for analytic connections, since he felt it was a special case of the general problem. But Narasimhan was impressed and because of his insistence, Raghunathan quickly wrote it up; the paper was well received and became a major part of his thesis.

The problem of deformations on special spaces like locally symmetric spaces leads naturally to questions on rigidity (=no deformations) of discrete subgroups of Lie groups. Raghunathan proved, in the latter part of his thesis and subsequent papers, a vanishing theorem for the first cohomology of (co-compact) discrete groups, with coefficients in local systems arising from representations of the ambient semi-simple Lie group. A special case (for the adjoint representation) is the Selberg–Weil local rigidity theorem. The vanishing theorem of Raghunathan had made a deep impression on many mathematicians and was indeed the precursor to several of his works, as well as work at TIFR by younger colleagues.

The topic of rigidity of discrete subgroups of Lie groups became Raghunathan's major preoccupation in his postdoctoral work.

Research contributions

Raghunathan's work on the cohomology of discrete groups broke new ground in the sixties.

Especially noteworthy was his proof that arithmetic subgroups of real Lie groups enjoy various finiteness properties. This was proved in a clever way by constructing a Morse function on the associated locally symmetric space and thereby showing that the locally symmetric space was the interior of a compact manifold with boundary. In particular, it implies that the cohomology groups of an arithmetic group are finite dimensional, which in turn proves that certain spaces of automorphic forms are finite dimensional.

This motivated two great mathematicians, Borel and Serre, to prove a more general result for what are called *S*-arithmetic groups, which they achieved by further algebraic constructions, but whose underlying motivation perhaps was to extend Raghunathan's proof to a more general algebraic setting.

He also proved a cohomology vanishing (in degree one) result for what was, at that time, the more difficult case of non-uniform arithmetic (higher rank) lattices; on the way, he discovered an interesting 'extension property' for these representations. The extension property states that given a finite dimensional representation of the arithmetic subgroup of a semi-simple Lie group, it extends (under some mild and natural restrictions) to a representation of the ambient semi-simple group (this extension property was proved in great generality using ergodic theoretic methods by Margulis (the Margulis super-rigidity theorem), who completely turned the argument around and proved the arithmeticity of higher-rank lattices using this extension property; this result was prominently cited in Margulis' Fields medal laudation).

By a different method, Raghunathan himself had proved a substantial part of the arithmeticity theorem of Margulis (i.e. in the cases when the Q-rank of the lattice in at least two). The arithmeticity result shows that contrary to the case of $SL_2(\mathbb{R})$, all lattices (discrete subgroups in (semi-simple) Lie groups with finite co-volume) in higher-rank Lie groups can be classified, up to finite index, as those coming from number theory.

Now arises the problem of classifying the finite index subgroups of higher-rank arithmetic groups. A conjectural solution to this problem (by Serre) essentially indicates (the 'congruence subgroup problem' or CSP) that all finite index subgroups of higher-rank arithmetic

Raghunathan and his family (son, daughter in law and grandchildren).

groups are 'congruence subgroups', up to a finite amount of discrepancy. The discrepancy can be measured as the size of a finite abelian group and is called the 'metaplectic kernel'.

Raghunathan's contributions to the congruence subgroup problem constitute a most impressive body of work, and have proved to be highly influential in the area. At present, his theorems are the most general results on the CSP. He proved Serre's conjecture in all cases when the lattice does not have a compact fundamental domain (the so-called non-co-compact case). In a series of papers with Gopal Prasad, Raghunathan also computed the metaplectic kernel, thus almost completely settling the CSP.

More recently, Raghunathan has done excellent work in collaboration with Lubotzky and Mozes, on comparison of the word metric and the Riemannian metric on a class of groups, which in particular settles a long-standing conjecture of Kazhdan. In a recent paper Raghunathan settled an interesting case of a conjecture of Grothendieck and Serre, on principal bundles, extending a result of Colliot-Thélène and Ojanguren.

Teacher and educator

Raghunathan has also played a major role as a promoter of mathematics, in addition to being a successful researcher. He has played an important part in the growth of the School of Mathematics at TIFR, as an international centre of excellence. He has given over the years, many high-level courses which have been instrumental in shaping the mathematical interests of many members of the School, both junior and senior. In particular, he has given many graduate-level courses which have been tremendously helpful to the students. He has vast knowledge and insights on various branches of mathematics (not all of it from book learning; he seems to learn better from lectures and seminars, often supplying the missing proofs and ideas himself). His willingness to share his insights and discuss freely so many areas of mathematics, and his enthusiasm and ability to communicate mathematics to others have been invaluable to members of the mathematical community.

Among his students are Dani, Gopal Prasad and Vinay Deodhar, who have become important researchers in their own right. Each of them has worked, during their thesis days, in different fields, and this reflects the breadth of Raghunathan's research interests as well. He has built a school in Lie theory and discrete groups which has acquired international reputation.

Discussions with Dani and a re-interpretation of what he had proved resulted in the 'Dani–Raghunathan conjecture', which was the driving force for further research in the area of dynamics of unipotent flows on the space of lattices. This was eventually completely solved by Marina Ratner (who also proved what she called 'Raghunathan measure conjecture'). The conjecture states that the closure of *any* one parameter unipotent group acting (linearly on $\mathbb{R}^n$ and hence) on the space of unimodular lattices in $\mathbb{R}^n$ is the orbit of a (closed) subgroup of $SL_n(\mathbb{R})$. Raghunathan and Dani recognized that this conjecture held the key to many questions on lattices and related questions on number theory, in particular the Oppenheim conjecture (this was eventually solved by Margulis using the dynamical interpretation provided by Raghunathan).

Gopal Prasad, who was then a student of Raghunathan, proved the strong rigidity theorem of Mostow in the case of non-compact ($\mathbb{Q}$-rank one) lattices, and also proved strong rigidity for very general classes of lattices. In a long and fruitful collaboration, Gopal Prasad and Raghunathan proved several fundamental results on the computation of what is called the 'metaplectic kernel'; the computation of this kernel is a big part of the congruence subgroup problem.

Deodhar proved several crucial results on generators and relations for quasi-split groups which were important in the computation of the congruence subgroup kernel.

From the above, it is clear that Raghunathan was able to successfully isolate important parts of his research programme and inspire his students (with material contributions from them, of course) towards the solution; I am sure that this benefited the students as well.

Raghunathan and Ramanan had organized a summer school in differential geometry in Mysore in 1980. The summer school was instrumental in shaping the interests of many of the students who attended it. I (and many of my fellow students) especially remember the inspiring lectures by Ramanan and Raghunathan.

Raghunathan's book *Discrete Subgroups of Lie Groups* (Springer) is a classic and has remained the standard reference on the subject. There have been developments on many of the questions raised in the book, but it still remains the fundamental foundational textbook.

Mathematical administrator and organizer

Raghunathan has also been active as an administrator and a promoter of research-level mathematics in the country. He was

Raghunathan receiving 'Padma Bhushan' from the President of India.

one of the founder members of the National Board of Higher Mathematics (NBHM), which has been instrumental in funding libraries, research fellowships, research project grants, and the mathematics departments of many institutes of higher learning. Many of the policies and aims of NBHM were formulated by Raghunathan and M. S. Narasimhan.

One of Raghunathan's most impressive achievements was the organization of the ICM in 2010 at Hyderabad. The logistics of hosting thousands of mathematicians from abroad and India, and dealing with administrative problems arising from them seemed formidable to me; I have seen him completely relaxed and seemingly unhurried during two to three years prior to and after the ICM (although, of course, completely immersed in its organization); he seemed to effortlessly deal with its organization. To be sure, he had the help of many colleagues dedicated to making the ICM a great success (which it certainly was), but even to inspire such dedication was a tremendous achievement. He was also instrumental in getting funding and Government support for such an event, and also managed to get material help from the Infosys Foundation. Many graduate students at Hyderabad have mentioned that their motivation to pursue a career in mathematics was because of the ICM.

He is presently heading the National Centre for Mathematics, a venture somewhat analogous to 'Oberwolfach', which has organized successful instructional schools and high-level mathematical conferences. It is housed in IIT, Bombay and is funded both by IIT and TIFR.

Raghunathan will be turning 75 in August 2016. He continues to be active, and has a great potential left in him both in terms of mathematical creativity and the ability to bring about progress in setting up mathematical research institutions. I wish him all the best.

Appendix 1. Mathematical terms used in the text

Here we explain briefly, some of the mathematical terms used in the text.

We will consider closed subgroups of groups like $SL_n(\mathbb{R})$, the group of $n \times n$ matrices with entries in $\mathbb{R}$. This comes equipped with the subspace topology induced from the natural topology on $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$, the space of $n \times n$ matrices with real entries. A subgroup of the form $SL_n(\mathbb{Z})$ is a discrete subgroup. Moreover, there is a natural volume form on $SL_n(\mathbb{R})$ invariant under both left and right translations of $SL_n(\mathbb{R})$. Hence the Hausdorff quotient $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ also has an $SL_n(\mathbb{R})$-invariant measure on it. It is a remarkable fact, proved by Hermite and Minkowski, that this volume is finite. Thus $SL_n(\mathbb{Z})$ is said to have finite covolume in $SL_n(\mathbb{R})$ and is discrete. Such discrete subgroups are called lattices.

Moreover, $SL_n(\mathbb{Z})$ comes naturally as a group of integer matrices and is a prototype example of an 'arithmetic' group. More generally, if $G \subset SL_n(\mathbb{R})$ is an algebraic subgroup (i.e. is the set of zeros of some polynomials on $M_n(\mathbb{R})$ with rational coefficients and is a subgroup of $SL_n(\mathbb{R})$), such that $G(\mathbb{R})/G(\mathbb{Z})$ carries a finite $G(\mathbb{R})$ invariant volume, then $G(\mathbb{Z})$ is said to be an 'arithmetic lattice'.

Consider the natural projection map $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/m\mathbb{Z})$, where $m \neq 0$ is an integer. The target $SL_n(\mathbb{Z}/m\mathbb{Z})$ is a finite group and is called a congruence quotient. The kernel of this projection is clearly the group of integral matrices which are congruent, modulo m to the identity matrix; such a group is called the 'principal congruence subgroup' of level m. Any subgroup of $SL_n(\mathbb{Z})$ which contains such a principal congruence subgroup $SL_n(m\mathbb{Z})$ is also called a congruence subgroup. The congruence subgroup problem asks if every finite index subgroup is a congruence subgroup.

The group $SL_2(\mathbb{Z})$ has many finite index subgroups which are not congruence subgroups. However, for $n \geq 3$, it is indeed true that every finite index subgroup is a congruence subgroup. The general question is this. Suppose G is a semi-simple group (essentially a closed connected subgroup of $SL_n(\mathbb{R})$ which does not contain any abelian normal subgroup) whose intersection $G(\mathbb{Z})$ with $SL_n(\mathbb{Z})$ is a lattice in G, is it true that every finite index subgroup of $G(\mathbb{Z})$ contains a congruence subgroup, i.e. a subgroup of the form $G \cap SL_n(m\mathbb{Z})$ for some m?

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