A glimpse of shape optimization problems

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In this mini review, we give a glimpse of a branch of geometric analysis known as shape optimization problems. We introduce isoperimetric problems as a special class of shape optimization problems. We include a brief history of the isoperimetric problems and give a brief survey of the kind of shape optimization problems that we (with our collaborators) have worked on. We discuss the key ideas used in proving these results in the Euclidean case. Without getting into the technicalities, we mention how we generalized the results which were known in the Euclidean case to other geometric spaces. We also describe how we extended these results from the linear setting to a non-linear one. We describe briefly the difficulties faced in proving these generalized versions and how we overcame these difficulties.

Keywords: Comparison principles, isoperimetric problems, moving plane method, maximum principles.

Introduction

AS mentioned in Anisa and Aithal[1], the following questions arise naturally from what we see around us. Why are soap bubbles that float in air approximately spherical? Why does a herd of reindeer form a round shape when attacked by wolves? Of all the geometric objects having a certain property, which ones have the greatest area or volume; and of all objects having a certain property, which ones have the least perimeter or surface area? These problems have been stimulating mathematical thought for a long time now.

Mathematicians have been trying to answer the above questions and this has led to a branch of mathematical analysis known as 'shape optimization problems'. A typical shape optimization problem is, as the name suggests, to find a shape which is optimal in the sense that it minimizes a certain cost functional while satisfying given constraints. Mathematically speaking, it is to find a domain Ω that minimizes a functional $J(\Omega)$, possibly subject to a constraint of the form $G(\Omega) = 0$. In other words, it involves minimizing a functional $J(\Omega)$ over a family $\mathcal{F}$ of admissible domains Ω. That is, the goal is to find a domain $\Omega^* \in \mathcal{F}$ such that

$$J(\Omega^*) = \min_{\Omega\in\mathcal{F}} J(\Omega).$$

In many cases, the functional being minimized depends on the solution of a given partial differential equation defined on a varying domain.

Isoperimetric problems form a special class of shape optimization problems. A typical isoperimetric problem is to enclose a given area $A > 0$ with a shortest possible curve. Here, $J(\Omega)$ = 'perimeter' of Ω and $G(\Omega)$ = ('area' of $\Omega) - A$. The definitions of 'perimeter' and 'area' depend on the ambient mathematical space. The classical isoperimetric theorem asserts that, in the Euclidean plane, the unique solution is a circle. This property of the circle is expressed in the form of an inequality called the isoperimetric inequality which is stated as follows: For any piecewise smooth simple closed curve C in a plane with arc-length l and enclosing area $A > 0$, we have

$$\ell^2 \geq 4\pi A,$$

and equality holds if and only if C is a circle of radius $\sqrt{A/\pi}$. To read more about the isoperimetric inequality, see Anisa and Aithal[1].

The first proof of the classical isoperimetric problem, as is recalled in Ritoré[2], is believed to be due to Zenodorus, who had written a treatise on isoperimetric figures. This treatise is lost but is known through the fifth book of the *Mathematical collection* by Pappus of Alexandria. He proved that among all polygons enclosing a given area, the regular ones have the least possible length. This implies the isoperimetric problem using the standard approximation argument. Since then, many proofs have been given, some of them incomplete, although employing interesting and fruitful ideas. Without even trying to be exhaustive, the list of mathematicians that have considered the classical isoperimetric problem include Euler, the Bernoulli brothers, Gauss, Steiner, Weierstrass, Schwarz, Levy and Schmidt.

Pappus shares some of the ideas of Book V by describing how bees construct honeycombs. His conclusion about honeycombs and the aims of his work are verbatim as follows: 'Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. But we, claiming a greater share in wisdom than the bees, will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always the greater, and the greatest of them all is the

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circle having its perimeter equal to them.' This also finds a mention in Kiranyaz *et al.*[3].

There are many other results similar to the isoperimetric inequalities wherein extrema are sought for various quantities of physical significance such as the energy functional or the eigenvalues of a differential equation. They are shown to be extremal for a circular or spherical domain. The Faber–Krahn Theorem[4–6] is an example of such a result. It states as follows: Let $\lambda_1(\Omega)$ denote the first Dirichlet eigenvalue of the Laplacian on a bounded domain Ω in $\mathbb{R}^n$. Then $\lambda_1(\Omega) \geq \lambda_1(B)$, where B is a ball in $\mathbb{R}^n$ such that $Vol(B) = Vol(\Omega)$, and equality holds if and only if $\Omega = B$. Please see Anisa and Aithal[1] for many important references on this topic. There is a plethora of examples of problems of this type wherein one tries to find the extremum of eigenvalues of elliptic operators. See Henrot[7] for more details. Henrot[7] also describes tools to solve these problems and talks about many open problems in this field.

Most histories of the isoperimetric problem begin with its legendary origin in what is called the Problem of Queen Dido. Her problem (or at least one of them) was to enclose an optimal portion of land using a leather thong fashioned from ox-hide. If Dido's was the true original isoperimetric problem, then what is needed is a solution not on the plane but on a curved surface. For the history of consideration of the isoperimetric problem on curved surfaces, refer Anisa and Aithal[1]. A survey up to the year 1978 of the isoperimetric problem on general Riemannian manifolds available in Ossermen[8] gives more detailed historic facts and is about developments in the theory of isoperimetric inequalities. This survey recounts many sharpened forms, various geometric versions, generalizations, and applications of this inequality. Again, see the introduction section of Anisa and Aithal[1] for other general references.

In this article, we define two standard functionals associated to the Laplacian. We will then discuss the shape optimization problems related to these functionals. We start with the results known in the Euclidean case and describe the key ideas involved in the proof. We then state its generalizations to other geometric spaces. We also describe how these results extend from the linear problems, viz. problems (1) and (2), to non-linear ones, i.e. from these problems involving the Laplacian to more general problems involving the p-Laplacian. Please see eq. (3) for the definition of the p-Laplacian. Notice that, for $p = 2$, the p-Laplacian is nothing but the Laplacian. We will describe briefly the difficulties faced in proving these generalized versions and how we overcame these difficulties.

The energy functional and the first Dirichlet eigenvalue of the Laplacian

Let $\Delta f = \mathrm{div}(\nabla f)$ be the Laplace Beltrami operator on a Riemannian manifold (M, g). Let B_0 and B_1 be open (geodesic) balls in (M, g) such that $\overline{B}_0 \subset B_1$. Let $\Omega := B_1 \backslash \overline{B}_0$. Consider the following equations

$$-\Delta u = 1 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega; \tag{1}$$

and

$$-\Delta u = \lambda u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{2}$$

Please refer to Aubin[9] (Theorem 4.4, p. 102) for a proof of the following theorem.

Theorem 1. *The eigenvalues of the (positive) Laplacian $-\Delta$ are strictly positive. The eigenfunctions corresponding to the first eigenvalue $\lambda_1(\Omega)$, are proportional to each other (i.e. the eigenspace is of dimension 1). They belong to $C^\infty(\overline{\Omega})$ and they are either strictly positive or strictly negative. Moreover,*

$$\lambda_1(\Omega) = \inf\{\|\nabla\phi\|^2_{L^2(\Omega)} \mid \phi \in H^1_0(\Omega) \text{ and } \|\phi\|_{L^2(\Omega)} = 1\}.$$

Let $y_1 := y_1(\Omega)$ denote the unique solution of problem (1), corresponding to $\lambda_1 := \lambda_1(\Omega)$, characterized by

$$y_1 > 0 \text{ on } \Omega \text{ and } \int_\Omega y_1^2 \mathrm{d}x = 1.$$

For $M = \mathbb{E}^n$, the Euclidean space of dimension n, Hersch[10], Ramm-Shivakumar[11] (for $n = 2$) and Kesavan[12] (for general n) proved the following:

Theorem 2. *If u is a solution of (1), the energy functional*

$$\int_{B_1\backslash\overline{B}_0} \|\nabla u\|^2 \mathrm{d}x,$$

attains its minimum if and only if B_0 and B_1 are concentric.

Theorem 3. *The first eigenvalue $\lambda_1(\Omega)$ of problem (2) attains its maximum if and only if the balls are concentric.*

Key steps in the proof

The shape calculus and the moving plane method are the key steps that are used to prove these theorems.

Shape calculus

Under the action of a suitable vector field V, a domain Ω in the admissible family $\mathcal{F}$ gets transformed to another domain in the same family. Let Φ_t denote the one parameter family of diffeomorphisms corresponding to this vector field V. Let $\Omega_t := \Phi_t(\Omega) \in \mathcal{F}$. Let $y_1(t) := y_1(\Omega_t)$, $y_1^t := y_1(t) \circ \Phi_t$ and $\lambda_1(t) := \lambda_1(\Omega_t)$. The shape calculus deals with the study of the behaviour of the following maps: $t \mapsto y_1(t)$, $t \mapsto y_1^t$, and $t \mapsto \lambda_1(t)$.

We have the following results. Please refer to Sokolowski and Zolesio[13], and Anisa and Aithal[14] for proofs.

Proposition 1. *The map* $t \mapsto (\lambda_1(t),\ y_1^t)$ *is a* $\mathcal{C}^1$*-curve in* $\mathbb{R} \times (H^2(\Omega) \cap H_0^1(\Omega))$ *from a neighbourhood of* 0 *in* $\mathbb{R}$.

Proposition 2. *The map* $t \mapsto y_1(t)|_{\Omega'}$ *is differentiable in* $H^1(\Omega')$ *at* $t = 0$ *and the derivative* y_1' *satisfies*

$$\left.\begin{aligned} y_1' &= \dot{y}_1 - g(\nabla y_1,\ V) \text{ in } H^2(\Omega), \\ y_1'|_{\partial\Omega} &= -\frac{\partial y_1}{\partial n} g(V, n). \end{aligned}\right\}.$$

Here, n denotes the smooth outward unit normal to $\partial\Omega$.

Proposition 3. $y_1' \in \mathcal{C}^\infty(\bar{\Omega})$.

Proposition 4 (Hadamard perturbation formula). *Let n denote the smooth outward unit normal to* $\partial\Omega$. *Then,*

$$\lambda_1' = -\int_{\partial\Omega} \left(\frac{\partial y_1}{\partial n}\right)^2 g(V, n)\mathrm{d}S.$$

Since λ_1 is invariant under isometries of Ω, it is enough to consider the family of domains $\Omega = B_1 \backslash \bar{B}_0$ where B_1 is static and is centred at the origin while the centre of B_0 is free to move along the positive x_1 axis. Since $\lambda_1(t)$ is a differentiable and even function, we get $\lambda_1'(0) = 0$. That is, the configuration where the balls are concentric serves as a critical point for $\lambda_1(t)$.

The moving plane method

Let B_0 be centred at $q(t) := (t, 0, 0, \ldots, 0) \in \mathbb{R}^n$. Let H be a hyperplane passing through $q(t)$ and orthogonal to the x_1-axis, that is, the line joining the centers of B_1 and B_0. Then, whenever $t > 0$, that is, when B_0 and B_1 are not concentric, H is a hyperplane of interior reflection. That is, H divides Ω into two unequal components such that the reflection of the smaller one, call it $\mathcal{O}$, about H is completely contained in the larger component. See Harrell *et al.*[15] for more shape optimization problems where a hyperplane of interior reflection plays an important role. Using the celebrated moving plane method which involves a reflection technique[16–18], we split the integrand in the Hadamard perturbation formula into two parts as

$$\begin{aligned} \lambda_1' &= -\int_{\partial\Omega} \left(\frac{\partial y_1}{\partial n}\right)^2 g(V, n)\mathrm{d}S \\ &= \int_{x \in \partial B_0 \cap \partial\mathcal{O}} \left\{\left(\frac{\partial y_1}{\partial n}(x)\right)^2 - \left(\frac{\partial y_1}{\partial n}(x')\right)^2\right\} g_x(V, n)\mathrm{d}S. \end{aligned}$$

Here, x' denotes the reflection of x about the hyperplane H.

We observe that the inner product of V and n has a constant sign almost everywhere on $\partial B_0 \cap \partial\mathcal{O}$. Now, as in Kesavan[12], using the strong maximum principle and the Hopf lemma from Protter[19], we prove that $\lambda_1'(t) < 0$ for $t > 0$. This proves Theorem 3. The proof of Theorem 2 is similar and easier to arrive at.

Generalizations

Space forms: Consider the unit sphere $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}$ with induced Riemannian metric $\langle,\rangle$ from the Euclidean space $\mathbb{R}^{n+1}$. Also consider the hyperbolic space

$$\mathbb{H}^n := \left\{(x_1,\ x_2, \ldots,\ x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^n x_i^2 - x_{n+1}^2 = 1 \text{ and } x_{n+1} > 0\right\},$$

with the Riemannian metric induced from the quadratic form $(x,\ y) := \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1}$, where $x = (x_1,\ x_2, \ldots, x_{n+1})$ and $y = (y_1, y_2, \ldots, y_{n+1})$.

The Euclidean space $\mathbb{E}^n$, the unit sphere S^n, and the hyperbolic space $\mathbb{H}^n$ defined above constitute what are called as the space forms (complete simply connected Riemannian manifolds of constant sectional curvature). Theorems (2) and (3) are true on the space forms as well (see Anisa and Aithal[14]).

Rank one symmetric spaces of the non-compact type: We generalize these results now from the space forms to what are called as rank one symmetric spaces of the non-compact type[20]. Consider the division algebra of the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. Consider the hyperbolic spaces of dimension n over the following division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Here, $\mathbb{R}$ and $\mathbb{C}$ respectively denote the usual real and complex fields. In the case of the octonions, we only consider $n = 2$ and the corresponding hyperbolic space is called the Cayley plane. These examples constitute a class of Riemannian manifolds referred to as rank one symmetric spaces of the non-compact type.

p-Laplacian: We describe how the above results extend from the linear problems, viz. eqs (1) and (2), to nonlinear ones, i.e, from problems involving the Laplacian to more general problems involving the p-Laplacian. The p-Laplacian (Δ_p) operator is defined as

$$\Delta_p u := \mathrm{div}(\|\nabla u\|^{p-2} \nabla u) \ \ (1 < p < \infty). \tag{3}$$

We will describe briefly the difficulties faced in proving these generalized versions and how we overcame these difficulties.

Generalization to space-forms

We developed shape calculus for general Riemannian manifolds[14]. The moving plane method works here as it does in the Euclidean case, because reflection in a hyperplane is an isometry in any space form, and so it commutes with the Laplacian.

Generalization to rank one symmetric spaces of the non-compact type

In a joint work with Vemuri[20], we considered this case. Since all non-compact rank-one symmetric spaces are what are called Damek–Ricci spaces (refer to Anisa and Vemuri[20] for a definition of Damek–Ricci spaces), we considered a special geometric configuration of balls in a Damek–Ricci harmonic space wherein the isometry group is doubly transitive. Therefore, it suffices to consider the family of domains in which the centres of B_0 and B_1 are on certain one-dimensional 'axis'. If for a simply connected Riemannian manifold, every hyperplane reflection commutes with the Laplacian, then it must have constant sectional curvature. Therefore, the moving plane method does not work. But a careful analysis of the proofs reveals that the commutativity of the Laplacian and reflection is needed only on a class of appropriately radial functions. A more elegant way is to work with isometries of the symmetric space, viz. the geodesic symmetries with respect to a point.

Generalization to a non-linear operator, namely, the p-Laplacian

In a joint work with Rajesh Mahadevan[21], we follow the same line of ideas as before. We studied the sign of the shape derivative using the moving plane method. Then, we developed and used various comparison principles instead of maximum principles. Carrying out this programme involved several technical difficulties. In the process, we obtained some interesting new side results like: (a) the Hadamard perturbation formula for the energy functional for the Dirichlet p-Laplacian; (b) the existence and uniqueness result for a nonlinear partial differential equation; and (c) some extensions of known comparison results for nonlinear partial differential equations with non-vanishing boundary conditions.

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