The importance of metaphorical thinking in the teaching of mathematics

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This communication assesses the relevance of metaphorical thinking in helping to develop students' comprehension of mathematical concepts. It is of utmost importance to examine the role of metaphors in the understanding of mathematical concepts as well as how different types of metaphors can provoke interpretations of mathematical problems for students before looking at the relevance of metaphorical thinking in the teaching of mathematics. This acts as a basis for grasping the importance of metaphorical thinking in fostering the transition of mathematical perceptions to theory and its applicability to day-to-day life.

Keywords: Mathematical concepts, metaphors, thinking, teaching.

MANY mathematics teachers and learners are typically left with the dilemma of trying to visualize mathematical objects in real-life situations. These mathematical objects may not exist in real life, but we can make models from soil, metal, wood, plastic and so on to represent them. For instance, a circle can be represented by a metallic ring. Mathematical metaphors, which are readily available in real life, can be selected and used to illustrate a mathematical concept with ease. By mathematical concept, we refer to the main idea of a mathematics problem, equation or formula. A creative and innovative teacher has the task of presenting to students different experiences from which the latter may generalize mathematical concepts and turn them into their own perceptions. This communication states how the metaphoric approach is the principal and possibly the single most powerful tool for the elucidation of mathematical concepts.

Metaphors remain crucial components of thinking[1]. The relevance of body-related metaphors in mathematical thinking has been clearly stated by Lakoff and Nunez[2], and Nunez *et al.*[3] with some examples concerning, in particular, natural numbers and continuity. We believe that no single metaphor is sufficient for addressing all aspects of a given concept. As a teacher, one is therefore faced with the task of selecting a series of metaphors that will enable one to represent all aspects of a mathematical concept.

A metaphor can be defined as a figure of speech in which a word or phrase literally denoting one kind of object or idea is used in place of another to suggest a likeness or analogy between them[4]. Any metaphor one constructs will have both ground and tension[5]. The ground of a metaphor constitutes the similarities between the objects involved in the metaphor, whereas the tension constitutes the dissimilarities between the objects in play. Therefore, the ground of the metaphor includes the aspects of both objects that allow one to observe phenomena that might otherwise remain unseen. The tension of the metaphor comes as a result of the fact that the metaphor is seen as a mere physical object. These are both vital in provoking the mind of the learner regarding a mathematical concept.

Cope[6] attributed the use of metaphors to delight in learning and traced delight in metaphors to the enjoyment of the author's ingenuity over the immediate or vivid presentation of the principal subject. Most importantly, the metaphors should depict exactly the intended expressive and purposive functions they are being used for. A metaphor produces an enhancement of the subject and evokes various feelings depending on how it has been presented to the learners. It has also been stated in regard to 'figurative' language that a metaphorical expression has a meaning that is a transformation of its normal literal meaning and is a special case of a more general view.

Research shows that many authors have written about the role of metaphors in the teaching and learning of mathematics[3,5,8–10]. It has also been asserted that the cognitive structure for advanced mathematical thinking shares the conceptual structure of non-mathematical daily life-thinking[10] while considering the importance of the embodied cognition theory. The metaphorical projection is the main cognitive mechanism that permits the structuring of the abstract mathematical entities by means of corporal experiences. We interpret the metaphor as the comprehension of an object, thing or domain in terms of another. Metaphors create a conceptual relationship between an initial or source domain and a final or target domain, while properties from the first to the second domain are projected. In relation to mathematics, Lakoff and Núñez[2] distinguish two types of conceptual metaphors:

- Grounding metaphors: these relate a target domain within mathematics to a source domain outside them.
- Linking metaphors: these maintain the source and target domains within mathematics and exchange properties among different mathematical fields.

Bergeson[11] describes scaffolding as a metaphor for the teacher's provision of 'just enough' support to help students progress or succeed in each mathematical learning activity. According to Greenfield[12], scaffolding – as known in building construction – has five characteristics: it provides support; it functions as a tool; extends the range of the worker; allows the worker to accomplish a

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task not otherwise possible; and is used selectively to aid the worker where need be. The implication is that metaphors can clearly bring out different aspects of a given mathematical concept if used appropriately.

Due to the complexity of mathematical language, scholars note that misconceptions can arise from those mathematical registers that have general meanings in everyday language, but more precise meanings in mathematical language contexts[13–15].

Furthermore, even for terms with the same meaning in both everyday life and mathematical language, if the embedded mathematical concept is not thoroughly understood by a learner, further learning in more advanced mathematics could be impeded. Since the learners may be encountering a given context of mathematics for the first time, they may not know the particular meanings of most of the mathematical words and expressions that are under discussion, and teachers cannot restrict themselves to professional mathematical language and sometimes must use non-professional, everyday language for conveying mathematical meaning.

In such situations, if teachers' and students' understandings of the words or expressions used are different from each other's, especially when a word has a mathematical meaning that is different from the colloquial meaning accessible to the students, the possibility of miscommunication between teachers and students may arise and finally result in student misconceptions.

Metaphorical thinking is useful for depicting conceptual understanding of teaching and learning, and therefore the metaphors of teaching mathematics should reflect and represent the underlying beliefs that govern performance in the mathematics classroom[16–19]. Metaphors represent beliefs, and metaphorical thinking helps understand professional thinking as a mathematics teacher[20]. Metaphors also provide a powerful cognitive tool for gaining insight into a teacher's professional thinking[19].

The most important aspect is the conventional concept of teaching and learning that governs thoughts and everyday teaching performance in the mathematics classroom, down to the most routine details[1]. The conceptual understanding of the conventional concept of teaching and learning, which is metaphorical in nature, plays a significant role in structuring a teacher's experience[1]. Consequently, the traditional view of teaching and learning impacts the structural role of the teacher as the transmitter and controller, and the students' roles as learners and passive receivers and listeners. A metaphorical approach for teaching mathematics can play a central role in defining everyday performance in mathematics.

A total of 10 lecturers (six males and four females) and 82 students in undergraduate and postgraduate studies (36 males and 46 females) took part in this research. The age range was from 18 to 45 (mean = 24.88, standard deviation = 3.65) with the majority born in Indonesia (93.4%). The mode of participation was purely on a voluntary basis, and none of the participants received any pay or any other form of credit apart from a word of appreciation from the researchers.

The major tasks involved observing the teaching and learning activities in class and interviewing the lecturers and selected students during and after the lectures. This was to find how metaphorical thinking influences critical thinking skills among learners by adopting a teaching and learning model that develops higher-order thinking skills in teaching and learning mathematics. The students' level of involvement in class was observed and assessed in relation to the design of a project. The aim of the design was the creativity and innovativeness of the lecturers and students towards a given mathematical concept. Lecturers were requested to produce a brief that states the design goals, design requirements and programmatic needs.

Twelve sessions were organized into three meetings per week, one or two hours per session, devoted to observation, problem-solving and interviewing of the students and lecturers. Students were requested to provide a personal interpretation of a mathematical concept as well as solutions to the problem design, relating the ideas to a real-life situation and giving a brief about their level of understanding. In the course of the sessions, students worked individually and, in some cases, in groups. In particular, they were asked to solve various mathematical problems and also come up with similar designs to the situation in question in relation to everyday activities in real life and the physical environment. Using different metaphors, students were requested to explore the design problem. The extraction of concepts from various metaphorical sources helped them to reinterpret conventional design situations in a much better way. During the different sessions of the design task, students engaged in various activities, including sketches, drawings and logical presentations of the problem design. The last few minutes of each session were dedicated to the development of the design concept and arriving at a solution that would meet the initial design requirements. Ten lecturers assisted and guided the students along the different stages of the design process. Conclusions were drawn based on the level of participation, creativity and innovativeness, and assessment results from the lecturers. Each lecturer's discourse was interpreted in the way that a focus was put on the pattern that he/she used to deliver the mathematical concepts and message, and the communicative strategies[21]. The lecturers who participated in this study have been teaching mathematics in government-aided Indonesia universities for over five years. Different classes and topics in mathematics were considered for the research.

At the end of each lecture and the completion of the learning process, a survey on the use of metaphors and design creativity was conducted. The lecturers and students were requested to assess the creativity of their projects and the use of metaphors in understanding mathematical concepts. The questionnaire included 20

Table 1. Assessing the attributes of the models and metaphors used

Area of concern	Frequency of the rating 1	2	3	4	5	Mean	Standard deviation (SD)
Functionality of the model/metaphor			20	47	25	4.05	0.49
Innovativeness of the student		4	56	30	2	3.33	0.35
Clarity of the model/metaphor			13	60	19	4.07	0.34
Practicality of the model/metaphor			38	51	3	3.62	0.30
Value of the model/metaphor		1	40	44	7	3.62	0.41
Flexibility of the model/metaphor		2	55	33	2	3.38	0.32
Productivity		5	66	21		3.17	0.25
Applicability to real-life situations			10	70	12	4.02	0.24
Availability of the model/metaphor			14	75	3	3.88	0.17
Consistency of the model/metaphor		12	50	29	1	3.21	0.45

Table 2. Assessing the role that metaphors play in teaching and learning mathematics

Area of concern	Frequency of the rating 1	2	3	4	5	Mean	SD
Student's engagement in an efficient design process			30	38	24	3.93	0.58
Level of organization of the student		2	42	32	16	3.67	0.61
Conceptual thinking		6	49	27	10	3.45	0.59
Generalization of the problem		5	51	29	7	3.41	0.50
Critical thinking		11	62	15	4	3.13	0.44
Relates initial knowledge to current knowledge		12	46	33	1	3.25	0.47
In line with the design objectives		20	40	27	5	3.18	0.69
Analyses the problem from a wider perspective		25	50	9	8	3.00	0.72
Generates alternative designs		6	54	31	1	3.29	0.36
Produces the expected outcome		4	59	26	3	3.30	0.36

key areas of the attributes of the models/metaphors used and their role in higher-order thinking in mathematics. Each question was rated from 1 to 5 by the respondent based on his/her own judgment. Tables 1 and 2 illustrate the outcome of the questionnaire analysed using Microsoft Office Excel. Ten lecturers and 82 students were interviewed after the lectures.

There were no demographic data provided; so the learners and lecturers were able to give their opinions and views without any compromise. We first examined how students evaluated their own designs according to creativity and metaphors. Next, we evaluated the relevance of the metaphorical teaching of mathematics.

Twelve different lectures were observed from different universities (Bandung, Indonesia), with different levels of studies in various topics (Table 3). The names of the universities have been concealed for security purposes.

Analysis of the findings was performed in three steps.

Step 1: The level of students' participation in class was observed.

To enhance the level of participation of the students in mathematics lessons, the lecturers acted as study guides and left most of the tasks to be handled by the students themselves. Throughout the lectures, students were observed to be active in the development of metaphors and in creative thinking regarding the problem design. Students' involvements were greatly influenced by peer-to-peer interactions that promoted and improved productive contributions. Teacher interaction also helped counterbalance some of the unequal participation of the students.

Step 2: Analysing the students' development of mathematical metaphors to suit the problem design.

To provide greater insight into the way students attempted to connect new mathematical concepts to existing knowledge, the lecturers introduced new mathematical concepts and assigned the students tasks to develop their own mathematical metaphors to suit the problem design. The level of student participation was observed to be very high, but many students lacked the socio-cognitive capacity needed to discuss intensive mathematical concepts. Analysis of the development of the mathematical metaphors by the students focused mainly on their conceptual understanding, strategic competencies, adaptive reasoning and productive disposition. The theoretical framework of carrier[22], drawing a semiotic perspective on meaning-making and on a conceptual view of metaphors, provides an outline of some possible links between meaning and metaphor, and a discussion of the relationship between metaphors and mathematical models. The majority of students were able to develop

Table 3. Details of the lectures observed and monitored

Duration/time	Level of education	Number of students	Topic
Double lesson (120 min)	S2	26	Uniform continuity
Double lesson (120 min)	S1	47	Quadratics and derivatives of functions
Single lesson (60 min)	S2	25	Sequences
Double lesson (120 min)	S1	35	Set theory
Double lesson (120 min)	S1	40	Probability
Single lesson (60 min)	S1	50	Integration
Double lesson (120 min)	S2	24	Differentiability
Double lesson (120 min)	S1	41	Numerical Integration
Single lesson (60 min)	S1	48	Vector spaces
Single lesson (60 min)	S1	40	Area under a curve
Double lesson (120 min)	S2	37	Continuous differentiability
Single lesson (60 min)	S2	26	Convergence sequence

S1, Undergraduate programme; S2, Masters' programme.

mathematical metaphors and models that were in line with the problem design.

Step 3: Assessing the level of creativity included in the questionnaire.

Creativity relates to originality, deep and flexible knowledge in content domains and long-term periods of work and reflection. According to Tammadge[23], there is an urgent need for teachers of mathematics to identify, encourage and improve creative mathematical ability at all levels. According to him, creativity includes the ability to see new relationships between techniques and areas of application, and to make associations between possibly previously unrelated ideas. In assessing the level of thinking, self-evaluation was conducted, and the results of the interviews with lecturers and selected students were analysed for radical decisions that were made.

The use of metaphors in the teaching of mathematics creates relations to the concept in play by allowing explanations that can be understood by anyone within a short period of time. If all teaching of mathematics is performed using only non-metaphorical mathematics words, it would be abstract to process the knowledge and decipher what it all means. For instance, in one of the classes, a lecturer illustrated a metaphor for the concept of a 'curve is a set of continuous points which are near each one another', using the idea that a basketball traces its curved path point-wise when thrown. This cleared the misconceptions among the students. By relating the concept to a two-dimensional curve, the students were able to make radical conclusions about a curve.

According to Lakoff and Johnson[1], a metaphor may thus be a guide for action. This fact can be understood by looking at the application of curved surfaces in real-life situations. Many scientists and technologists have used the knowledge of curves to construct a number of structures, such as bridges, houses and discs, to mention a few. Therefore, metaphors can be used as master switches to change belief sets and teaching practices in mathematics[17].

Stimulating the learners' conscience in a broader perspective creates analogies in the form of metaphors between aspects of teaching and known mathematical concepts. Thus, changing the teaching metaphor can lead to alteration of the teaching and learning performance by raising the significance of classroom practice. The metaphors can stimulate reflective and critical thinking regarding a given mathematical concept. Perry and Cooper[9] note that metaphors are used as an educative tool for reflection.

Alternative teaching and learning metaphors help one to perceive learning and teaching from a different perspective. Alternative metaphors may provide fresh lenses through which prospective teachers become capable of viewing teaching and learning from different theoretical perspectives[19]. For instance, in case of minima and maxima, metaphors can be used to illustrate the concept of maxima. They help explain facts in a real situation. The same mathematical model/metaphor can be used to illustrate more than one mathematical concept. For instance, we can use metaphors to explain the trajectories of a ball in soccer and rugby. The angle of projection determines the range of the object. These two examples can also be alternative metaphors as one can easily consider trajectories of gun bullets, golf balls, water fountains and many others to represent the same mathematical concept.

During the implementation of a metaphorical approach in the teaching of mathematics, it is necessary to have a proper lesson plan that can facilitate metaphorical thinking to promote applied thinking and adaptations for diverse student needs. Without a well laid out lesson plan, content delivery may lack the sequential characterization of information, and this can cause a tremendous effect on the meaning of the metaphor being delivered. Giving students support at the beginning of a lesson and gradually equipping them to operate independently will help them to develop creative and higher-order thinking skills.

Based on this, Chiu (unpublished) states that metaphorical reasoning can facilitate computation, both through metaphorical computations and metaphorical constraints. This is a core to mathematical applications in many fields.

According Chiu[24], students can use metaphorical reasoning to understand new concepts, connect mathematical ideas, improve recall, understand mathematical representations and enhance their computational environment.

According to Presmeg[5], it is important to understand teachers' spontaneous metaphors that occur during the course of instruction given that they can influence the thinking of the students. Therefore teachers play a central role in shaping the students' abilities and ways of thinking. This is easily facilitated by the metaphorical approach to teaching mathematics.

Wilson and Cooney[25] note that students learn mathematics most effectively when they construct meanings for themselves, rather than simply being told. A metaphorical approach to teaching helps students create these meanings and learn in a more constructive manner. This can be best done when the teachers lead and motivate students to make new findings.

From a broader perspective, a metaphorical approach to teaching mathematics enables students to acquire mathematical knowledge, comprehend it and apply it to real-life situations. Students will also be able to analyse, evaluate and think creatively and innovatively. It is important to note that the ability to think creatively is a process, and that it is the highest-order cognitive learning process that facilitates the development of creative thinking skills. The results of the analysis indicate that metaphorical thinking is a powerful tool in teaching mathematics, as it is a key to creative and critical thinking. It has also been discovered that metaphorical thinking develops students' ability to meet the criteria of practicality and applicability of the mathematical concept. Metaphorical thinking in the teaching of mathematics also enables students to use a mathematical concept according to their mindset, to give rise to diverse interpretations of the problem.

Although many students generally have negative attitudes toward mathematics, the metaphorical approach to the teaching of this subject exposes the main ideas and simplifies them to give students the motivation to approach mathematics more positively. When the metaphors are developed by students themselves, it demonstrates that they are reflectively and creatively thinking. In conclusion, the use of metaphors in the teaching of mathematics helps us reflect on our professional lives and practices from a diverse perspective. Thus, a metaphorical approach to teaching and learning mathematics encourages the process of reflection to be ongoing and purposeful.

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