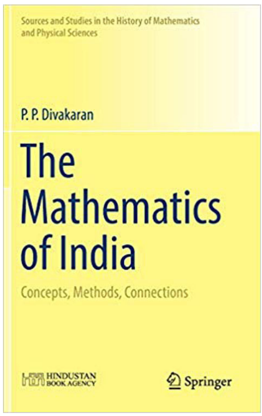

The Mathematics of India: Concepts, Methods, Connections. P. P. Divakaran. Hindustan Book Agency (India), P 19 Green Park Extension, New Delhi 110 016, India and Springer Nature Singapore Pte Ltd, 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore. 2018. xi + 441 pages. Price: Rs 980/74,96€.

While it is well known that India has a long and rich tradition in Mathematics, it is hard to come by books which explain the specific contributions in detail, trace the evolution and continuity of mathematical ideas, and survey the historical and social background in which research in mathematics was carried out. Divakaran's excellent book, which is readable, scholarly and well-researched, fills this need.

André Weil, in his lecture 'History of mathematics: why and how?' given at the International Congress of Mathematicians (1978) discusses the aim and content of a work on the history of mathematics. The first aim is to keep before us instances of first-rate mathematics, highlighting the mathematical ideas involved and their interconnections. A biographical sketch bringing alive mathematicians, their environment, in addition to their writings would be desirable. Weil says that 'An indispensable requirement is an adequate knowledge of the language of the sources; it is a basic and sound principle of all historical research that a translation can never replace the original when the latter is available.' It is important to give an exposition of the results and methods using modern mathematical notation, concepts and language. Weil also says that it is necessary not to yield to the temptation of concentrating only on the work of past great mathematicians, neglecting the work of lesser mathematicians. The book under review fulfills these requirements.

Incidentally, one of the reasons for a professional mathematician's interest in the history of mathematics is the hope that it may reveal hidden ideas which may be useful for further research. Thus the purpose of the history is 'to serve as an inspiration and promote the act of discovery' and not just for 'aesthetic enjoyment'.

The book deals mainly with three periods of mathematical tradition in India: (1) the ancient period beginning approximately in BCE 1200, (2) the 'Golden Age' spanning the fifth to twelfth centuries and (3) the period of the Kerala School of Mathematics, i.e. the fourteenth to sixteenth centuries. (The book refers to this third phase as the Nila School.)

The ancient period

The principles of a decimal enumeration system of numbers date from this period. The system was perfected over a millennium and evolved as the decimal place value system for denoting numbers, namely 'the ingenious method of expressing every possible natural number as a set of ten symbols (0, 1, 2,…, 9), including zero, each symbol having a place value'. The impact of this innocent looking invention cannot be overestimated in view of its utility in commerce as well in the development of mathematics in India and in Europe. Newton defined power series, and in particular polynomials, in one variable, in analogy with this 'new doctrine of numbers' and observed that these algebraic expressions in one variable can be manipulated (added, multiplied…) in the same way as 'common numbers'.

The main source of mathematical knowledge of this period is the *Sulbasutras* (rules of the cord), a manual for building ritual altars. The book under review contains a detailed analysis of the contents of *Sulbasutras*. The *Sulbasutras* deal with elements of plane geometry including the theorem of the diagonal (i.e. 'Pythagoras' theorem') and rectilinear figures and their transformations into one another with a given relationship between the areas of the figures. They also describe a geometric method for finding a good approximation to the radius of the circle whose area is that of a given square (and its converse construction).

Numbers like the square root of 2 and π were considered but the distinction between rational and irrational numbers does not seem to have been perceived. Indian mathematics never adopted the decimal representation of fractions throughout the course of history, as the author remarks. The conception of zero as a number in its own right as any other number (and not just a placeholder in the decimal system of enumeration) and its introduction into calculations count among the most original contributions to Mathematics from India. This topic is briefly discussed in section 5.3, 'Infinity and zero' of the book.

The Golden age

The second period, sometimes called the Golden Age of Indian Mathematics, lasted from the fifth to twelfth centuries CE and was dominated by the names of Aryabhata, Brahmagupta and Bhaskara II. Much of the mathematics during this period was closely related to astronomy (the study of the motion of celestial objects). A great contribution of this period was the development of plane trigonometry by Aryabhata. He defined the sine function and set up the difference equation for the sine function. This work of Aryabhata was very influential and was a major input in the work of the Kerala school of mathematics which will be described later.

A subject to which all these three mathematicians made fundamental contributions is the study of indeterminate equations (also called Diophantine equations). Here one seeks for integer solutions (not just rational solutions) of a polynomial equation whose coefficients are integers.

(One can also consider systems of polynomial equations of this type. This is a hard problem of much current interest. Think of Fermat's 'last theorem' concerning the integer solutions of the equation $x^n + y^n = z^n$!) Indeterminate equations of first and second degrees were studied during this period and this work received much acclaim as a high point of mathematical contributions from India.

The linear Diophantine equation of the type $ax + by = c$, where a, b, c are integers and x and y are integers to be determined, was essentially solved by an extension of the Euclidean algorithm. Such problems arose from astronomy.

Much more difficult to treat was the quadratic Diophantine equation

$$Nx^2 + 1 = y^2,$$

N being a positive integer, considered by Brahmagupta. (Centuries later this equation was named Pell's equation by Euler as a result of a misunderstanding. Pell was not the first person to notice this equation, nor did he find a solution.)

To deal with this problem, Brahmagupta considered the more general problem $Nx^2 + C = y^2$, where the integer C will be treated as an auxiliary parameter. He showed that if (x, y, C) is a solution of $Nx^2 + C = y^2$ and (x', y', C') is a solution of $Nx'^2 + C' = y'^2$ one can write down an explicit solution of $Nx^2 + CC' = y^2$.

(Thus he considered the set S of all solutions (x, y, C) for all values of C and defined a binary operation on S. Thus he defined a structure on the set of all solutions, in this case an algebraic structure, a very modern way of thinking. This operation has been called *Bhavana*.) Using this operation he found solutions for Pell's equation in some cases. While he could not solve the equation in general (which was done by Lagrange centuries later, when the integer N is not a perfect square), he found that he could solve the equation provided one can solve one of the equations

$$Nx^2 + C = y^2,$$

where $C = -1, 2, -2, 4, -4$.

Bhaskara's famous book *Lilavati* served as a basic textbook for generations of mathematicians to study arithmetic and geometry. It appears that Bhaskara wrote this book to teach mathematics to his daughter, a progressive act in an age when knowledge was passed on primarily from father to son.

His book *Bijaganita* also contains an algorithm, due to Jayadeva, to find a solution of Pell's equation, called *cakravala* or cyclic method. Starting from a known solution of $Nx^2 + C = y^2$, it sets up an algorithm to find a solution of $Nx^2 + C = y^2$, where $C = -1, 2, -2, 4, -4$. From here we can find a solution of Pell's equation, by the result of Brahmagupta alluded to above. However it seems that it was proved only in the nineteenth century that this algorithm yields the desired result.

This work is still of interest, as Pell's equation is related to quadratic number fields and binary quadratic forms. *Bhavana* is the manifestation of the multiplicative property of the norm in a quadratic number field and solutions of Pell's equation yield 'units' in a real quadratic field.

Brahmagupta had facility in dealing with negative numbers and stated the rule for multiplication of signs. This is remarkable since it took many more centuries for negative numbers to be accepted. Brahmagupta had some beautiful results on cyclic quadrilaterals, that is, quadrilaterals inscribed in a circle,

While much of mathematics during this period was driven by astronomy, the examples of quadratic Diophantine equations and cyclic quadrilaterals show that mathematics was also cultivated for its own sake.

The author gives a detailed account of the contents of Aryabhata's book *Aryabhatiya*, particularly the *Ganitapada* portion. Since *Aryabhatiya* is difficult to read, the author draws upon commentaries on the text, especially by Nilakantha. The 'kuttaka' method of solving linear Diophantine equation is explained. The ideas which go into the 'invention of trigonometry' are explained in sections 7.3 and 7.4.

For a clear exposition of the quadratic Diophantine problem, *Bhavana*, and the *Cakravala* one may refer to sections 8.2 and 8.3.

The Kerala (or Nila) school of mathematics

It was believed for some time that mathematical activity and creativity in historical India ceased in the twelfth century. As a matter of fact, during the fourteenth to sixteenth centuries there was a burst of mathematical activity in Kerala giving rise to what is arguably the finest of Indian mathematics. A small number of mathematicians living on the banks of the river Nila, in Kerala, constituted what is now known as the Kerala School of mathematics. (The author prefers to call this the Nila School to distinguish it from an earlier School in Kerala, see Chapter 9.) The founder of the School was Madhava. He and his School discovered the power series expansions for the functions sine, cosine and arctangent, and developed infinitesimal calculus for trigonometric functions, polynomials and rational functions. In particular the famous formula

$$\pi/4 = 1 - 1/3 + 1/5 - \ldots\ldots\ldots,$$

which was found by Gregory and Leibniz centuries later, was known to the Kerala School. This series is now known as the Madhava–Gregory series. Madhava has been compared with Newton and Leibniz, the discoverers of calculus in Europe.

The primary source for the path-breaking work of the School is *Yuktibasha*, by Jyeshthadeva. This text is written in Malayalam, and not in Sanskrit in which scholarly books used to be written. Divakaran, who can read Malayalam, says of the book: 'Motivations, conceptual inventiveness, technical advances (including proofs) are all given in a meticulous and sophisticated treatment in unambiguous Malayalam prose, a far cry from the enigmatic sutras of earlier masters.'

Divakaran has spent several years studying and researching material about the School. He writes knowledgeably, passionately and authoritatively about the members of the school, their work and their social background. The whole of part III of the book is devoted to a detailed and comprehensive description of the work of the School, wherever appropriate in modern mathematical language.

Divakaran has studied the continuing influences of the idea of recursion on Indian mathematics and believes it is one of the major features of *Indian Mathematics*. David Mumford says elsewhere that '*Yuktibasha* gives a unique insight into Indian methods: these are recursion, induction and careful passage to the limit'. All these three come together in the mathematics of the Kerala School.

For instance, the calculus for the sine function is built up by starting with Aryabhata's result on the difference equation for the sine function and passing carefully to the limit. This anticipates d'Alembert's dictum: 'The true metaphysics of infinitesimal calculus is nothing else than the notion of limit.'

It took more than two centuries for the work of this School to be recognized. As the author says, 'It has taken a long time for modern scholars to go from relative ignorance to puzzled admiration to an informed appreciation of the brilliance and originality of this achievement.'

Soon after the discovery of calculus by Newton and Leibniz, there was in Europe an explosion of calculus and its applications to natural sciences, powered by great mathematicians like Euler, Lagrange, Laplace, Cauchy and others. On the other hand, there was hardly any echo in India (and elsewhere) of the work of Kerala School. In fact, mathematics as a creative activity ceased to exist in India till modern times.

The rediscovery owes a great deal, on the one hand, to Indian mathematicians, K. Balagangadharan and C. T. Rajagopal among them, who studied this work and wrote (starting from the 1940s) expositions of the work in modern language, which brought this work to the attention of the international mathematical community. On the other hand (at around the same time) a critical edition of *Yuktibasha* was published by scholars. The name 'Kerala School' is now familiar to the general public in India. Could it be that the long delay in the recognition of the remarkable contributions of the school was due to the hegemony of Sanskrit, because *Yuktibasha* was written in the local language, Malayalam?

Conclusion

The last part, titled 'Connections' treats many topics which are relevant to a proper appreciation of the course and sociology of the development of mathematics and is not easy to summarize. It does not avoid dealing with vexed questions like priorities, originality, transmission of ideas, and the role of proof in Indian mathematics. These are questions which evoke much passion among historians of mathematics (and mathematicians); for instance some eurocentric mathematicians would denigrate Indian mathematics and some Indian mathematicians would exaggerate Indian contributions. The author analyses the different viewpoints and presents his own conclusions which are sensible and non-dogmatic.

He also discusses the role of faith in individual mathematicians. Some successors of Madhava and also his biological descendants are thought to have adhered to Lokayata philosophy, but about Madhava we do not have enough information to know if he also did.

Concerning the transmission of knowledge, it is striking that while Indian mathematicians were receptive to Greek astronomy, there was no influence of Euclid's 'Elements', and no trace of any of the following in Indian mathematics: prime numbers, prime factorization ('the fundamental theorem of arithmetic'), the treatment of incommensurables as in Euclid's 'Elements' and a familiarity with axiomatic and deductive methods. Other intellectual activities in India which might have been relevant for mathematics also had no influence. The author says: 'It is futile but fascinating nevertheless to ponder how a wholehearted adoption of Paninian structural methods might have transformed India's mathematical landscape.'

As for the transmission from India, Indian mathematics, especially Algebra, was studied and developed by the Arabs (a generic term which included inhabitants of present day Iran, Central Asia and some Arabic speaking countries) and transmitted by them to Europe. The development of Algebra in the sixteenth century in Italy, influenced by the mathematics originating in India and Islamic countries, started modern mathematics and the renaissance of mathematics in Europe.

The knowledge of the decimal place value system was also transmitted to Europe by the Arabs.

Exposition

The exposition in the book is tuned to the matter under discussion. Those with little mathematical background can get a gentle introduction to what natural numbers are (Peano axioms), and what recursion and induction mean. They can also learn about the decimal system of enumeration (section 4.2). Those with some mathematical background would enjoy reading in modern notation and mathematical language, how the power series expansion for the sine function was derived by the Kerala School (section 12.2). Even someone with no interest in mathematics or history of mathematics can read with pleasure (in sections 9.2 and 9.3) a fascinating social history of Kerala at a certain period of its history.

I have passed over other topics treated in the book, like mathematics in the Indus Valley civilization, the influence of Greek Astronomy, Jaina and Buddhist Mathematics, and the Bakhshali manuscript.

The book is highly recommended for anyone interested in understanding in depth the history of mathematics of India. While the material in certain sections is somewhat densely packed, reading these sections with close attention would be a rewarding experience,

M. S. NARASIMHAN

Department of Mathematics,
Indian Institute of Science,
Bengaluru 560 012, and
TIFR CAM, Bengaluru 560 065, India
e-mail: narasim@math.tifrbng.res.in

Annual Review of Medicine, 2018. C. Thomas Caskey, Mary E. Klotman and Peter Scardino (eds). Annual Reviews, 4139 El Camino Way, P.O. Box 10139, Palo Alto, California 94303-0139, USA. Vol. 69. vii + 506 pages. Price: US$ 112.

Annual Review of Medicine is a must read for those physicians who are inclined to keep track of advancing frontiers in medicine and thus be aware of changing paradigms in diagnostic and therapeutic approaches. The book offers lucidly composed, authoritative, critical and comprehensive, yet concise reviews on selected topics in a variety of specialities. The latest volume has 34 articles; 11 of them are on themes related to cancer and 9 deal with cardiovascular diseases.

Articles on cardiovascular diseases are about atrial fibrillation, heart failure, hypertension, hypercholesterolemia, sudden death after myocardial infarction, peripartum cardiomyopathy, heart disease in athletes and cardiovascular complications in patients with cancer.

Catheter ablation introduced two decades ago, is a requisite treatment method for atrial fibrillation. Efficacy, efficiency and safety of the procedure have bettered in recent times, thanks to improvement in strategy and techniques of ablation. These include focal impulse and rotor modulation (FIRM)-guided ablation, availability of force sensing catheters, automated atrial mapping and cryo ablation therapy. Rakesh Latchamsetty and Fred Morady from the University of Michigan review the indications for and the goals of ablation, describe the advances in the technologies and strategies, assess short and long-term outcomes after ablation and discuss a multidisciplinary approach for the long-term management of patients with atrial fibrillation.

J. D. Gladden and colleagues from Mayo Clinic, Minnesota survey the current knowledge of the epidemiological factors, clinical features and pathophysiology as well as the diagnostic approach and treatment options for heart failure with preserved ejection fraction (HFpEF), which is expected to be the most common form of heart failure in the coming decade. They conclude that diagnosis of HFpEF is a challenge and that it is necessary to discover newer diagnostic techniques and ways to