Nuclear Hulthén potential and the scattering phase shifts for $\ell = 3$

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Simple potential models of nuclear Hulthén type are proposed and parameterized to reproduce the nucleon–nucleon scattering phase shifts for the partial wave $\ell = 3$. The phase shifts are computed utilizing the phase function method for neutron–proton (n–p) and proton–proton (p–p) systems and compared with standard data to judge the merits of our models. Reasonable agreement in phase shifts is achieved with the results of more sophisticated calculations. Particularly, our models reproduce better results for n–p system over the p–p system.

Keywords: Nuclear potential, phase function method, scattering phase shifts, partial wave.

THE studies on neutron–nucleon (n–n) systems offer plenty of reliable experimental data[1–14]. They are rather correct for the proton–proton (p–p) system, whereas slight uncertainties are associated with the neutron–proton (n–p) system[15]. Therefore, a theoretical description of the n–n scattering provides basic information for understanding the nuclear structure and scattering of two- and many-body systems[16,17]. One generally constructs an interaction in a phenomenological manner to reproduce the standard values of low-energy scattering parameters and phase shifts for a particular system. The phenomenological potentials have many free parameters to be fitted into the experimental scattering data and phase shifts. Their ability to describe n–n interaction, their flexibilities and convenience for using nuclear structure calculation is remarkable. The database regarding the analysis of low-energy result has been augmented considerably by a number of groups[18–27]. The low-energy scattering parameters presented by these groups do not differ much, although the methods are different. Earlier, we had constructed higher partial wave energy-independent and energy-dependent two-nucleon interactions from the ground state interactions in conjunction with the corresponding regular solution using the supersymmetry-inspired factorization method[28–31]. We also proposed a four-parameter nuclear Hulthén potential[32] and computed the related phase shifts for the partial waves up to $\ell = 2$ using the phase function method (PFM)[33].

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Nuclear Hulthén potential and PFM

Arnold and Mackellar[34] parameterized Hulthén potential that fits the binding energy of the deuteron and low-energy scattering parameters. A one-term nuclear Hulthén potential that produces attraction only cannot report the change of sign in the scattering phase shifts. To have both attraction and short-range repulsion in the nuclear part of the interaction, one must consider a two-term Hulthén potential[32]. We have computed phase shifts for both n–p and p–p scattering[28–32] and found that they agree well with those of standard data[18–27] in the low and intermediate range of energies. As the phase shifts of the sub-states ($^3F_2$, $^3F_3$, $^3F_4$ and $^1F_3$) for the partial wave $\ell = 3$ do not change their sign within the energy range 0–200 MeV, a one-term potential may be sufficient to reproduce the same. Here we propose both one- and two-term nuclear Hulthén potentials, and judge their merits by computing scattering phase shifts. The potential models are defined as

$$V_N^1(s) = -S_1 \frac{\mathrm{e}^{-\beta s}}{(\mathrm{e}^{-\alpha s} - \mathrm{e}^{-\beta s})}, \tag{1}$$

and

$$V_N^2(s) = -S_1 \frac{\mathrm{e}^{-\beta s}}{(\mathrm{e}^{-\alpha s} - \mathrm{e}^{-\beta s})} + S_2 \frac{e^{-(\alpha+\beta)s}}{(e^{-\alpha s} - e^{-\beta s})^2}. \tag{2}$$

The terms $V_N^1(s)$ and $V_N^2(s)$ represent the one- and two-term nuclear potentials respectively. The quantities S_1, S_2, α and β are the state-dependent strength and range parameters of the interactions. The first term in the two-term potential $V_N^2(s)$ is attractive in nature like $V_N^1(s)$ and the second part, which behaves like a centrifugal barrier for small values of s, is repulsive in nature. Generally, the higher angular momentum state potentials are developed by addition of appropriate centrifugal barriers and keeping this in view, the second part is suitably added.

For charged hadron systems, however, one has to add an electromagnetic interaction to the nuclear part of the potential. As an electromagnetic interaction a particular screened Coulomb potential, the atomic Hulthén potential, is used here. Thus, for p–p scattering the actual interaction is written as

$$V_P^1(s) = V_{0A}(s) + V_N^1(s), \tag{3}$$

Table 1. Parameters for one- and two-term potentials

Partial wave states	Interactions	S_1 (fm^{-3})	S_2 (fm^{-3})	α (fm^{-1})	β (fm^{-1})
$^3F_2$	One-term $V_N^{(1)}$	0.75		0.26	1.45
$^3F_3$		–1.45		0.62	1.58
$^3F_4$		1.12		0.68	2.05
$^1F_3$		–1.62		0.38	1.55
$^3F_2$	Two-term $V_N^{(2)}$	2.45	1.85	0.18	1.23
$^3F_3$		2.45	3.46	0.38	1.65
$^3F_4$		2.45	1.25	0.68	2.05
$^1F_3$		2.45	3.25	0.62	1.65

and

$$V_P^2(s) = V_{0A}(s) + V_N^2(s), \tag{4}$$

where

$$V_{0A}(s) = V_0 \frac{e^{-s/a}}{(1-e^{-s/a})}, \tag{5}$$

the Hulthén potential with the real parameters V_0 and a. When $a \to \infty$ and the strength $V_0 \to 0$ such that $aV_0 = 2k\eta$, where η the Sommerfeld parameter, is a constant quantity[35]. Screened and cut-off Coulomb potentials are frequently used in many branches of physics, such as atomic, nuclear, particle and plasma physics. Thus, most of the physical observables for the short-range potentials have to be customized for charged particle scattering. When the particles interact via the electromagnetic potential (Coulomb), they never behave like free particles and the standard scattering theory does not hold good. Therefore, within the formalism of non-relativistic scattering theory, infinitely long-range electromagnetic interaction is replaced by a short-range interaction, i.e. a screened Coulomb interaction. As the Coulomb potential is not a well-behaved potential, the concept of phase shift is not well-defined according to the standard scattering theory. Thus, the standard phase function method for the local potential is not valid for infinitely-long range potential. In such a case, one has to treat it separately. For small values of s, the Hulthén potential behaves like a Coulomb potential, and for large s, the potential decreases exponentially to have less number of bound states compared to the Coulomb potential[36]. Thus, to avoid the real problem in computing phase shifts, a short-range electromagnetic potential, viz. the Hulthén potential replaces the Coulomb potential.

The phase function method is a powerful tool for calculating phase shifts for quantum mechanical scattering with local[33] and nonlocal[37–39] interactions. In this method the radial wave function of the Schrödinger equation is separated into two parts: (i) an amplitude part $\alpha_\ell(k, s)$, and (ii) an oscillating part with variable phase $\delta_\ell(k, s)$. The phase of the wave function at every point is defined by the quantity $\delta_\ell(k, s)$, known as the phase function. The phase function $\delta_\ell(k, s)$ for a local potential $V(s)$ satisfies the equation

$$\delta'_\ell(k,s) = -k^{-1} V(s) \times [\hat{j}_\ell(ks)\cos\delta_\ell(k,s) - \hat{\eta}_\ell(ks)\sin\delta_\ell(k,s)]^2, \tag{6}$$

where $\hat{j}_\ell(ks)$ and $\hat{\eta}_\ell(ks)$ are Bessel functions of the Riccati type. The quantity $\delta'_\ell(k,s)$ denotes the derivative of $\delta_\ell(k,s)$ with respect to s. Henceforth, the phase convention of Calogero[33] will be followed here throughout with Riccati–Hankel function of the first kind defined as $\hat{h}^1_\ell(x) = -\hat{\eta}_\ell(x) + i\,\hat{j}_\ell(x)$. One gets the phase shift $\delta_\ell(k)$ when the phase equation is solved from the origin to the asymptotic region. Finally, one gets the phase shift $\delta_\ell(k) = \lim_{r\to\infty}\delta_\ell(k,s)$.

With the parameterized potentials defined in eqs (1)–(5), the phase shifts will be calculated using the PFM (ref. 33) for the partial waves $\ell = 3$ and the associated results reviewed. Table 1 gives the parameters for various partial wave states under consideration.

Results, discussion and conclusion

With the potentials defined in eqs (1) and (2), the phase shifts of the corresponding partial wave states under consideration along with the associated parameters in Table 1 are computed (Figures 1–4). Our results are compared with those of Arndt *et al.*[20] and Gross and Stadler[27]. Here we have chosen $a = 50$ au, $V_0 a = 0.03472$ fm^{-1} (ref. 39) and $\hbar^2/2m = 41.47$ MeV fm^2. The n–p and p–p phase shift values are designated as $\delta^1_{np}, \delta^2_{np}$ and $\delta^1_{pp}, \delta^2_{pp}$ for the one- and two-term potentials respectively.

In Figure 1, the phase shift values δ^2_{np} for $^3F_2$ state agree well with that of Arndt *et al.*[20] in the energy range 75–200 MeV but differ slightly between 25 and 75 MeV, while those for one-term potential, δ^1_{np} show the similar characteristics as those of the two-term potential up to 150 MeV and beyond, differing significantly from

experimental data[20]. On the other hand, δ^1_{np} and δ^2_{np} are comparable with those of Gross and Stadler[27] up to 125 MeV. For the p–p case, both δ^1_{pp} and δ^2_{pp} differ by 0.5°–1° from the standard data[20] over the entire range. In Figure 2, the $^3F_3$ phase shifts δ^2_{np} show better agreement with the those of Gross and Stadler[27], than the one-term counterpart δ^1_{np}, while δ^2_{pp} values are in close agreement with that of Arndt *et al.*[20]. However, the values for δ^1_{pp} are not quite effective in this case. For the $^3F_4$ state, the values of both δ^1_{np} and δ^2_{np} give good fit to experimental data[20], but have slightly higher values (by 0.25°) than those of Gross and Stadler[27]. On the other hand, the values for δ^1_{pp} and δ^2_{pp} gradually decrease from the standard data[20] as the energy increases up to a maximum of about 1°, but resemble the nature of phase shifts for p–p scattering. In Figure 4, the phase shifts δ^1_{np} and δ^2_{np} reproduce correct nature and numerical values, except at low energies. Also, δ^2_{np} values are more reliable than δ^1_{np}.

In Figures 5–8 we have plotted one- and two-term n–p, p–p potentials for various states. The nature of the potentials is fully consistent with the parameters given in Table 1, together with eqs (1)–(5). Figures 5 and 7 show that one-term $^3F_2$ and $^3F_4$ potentials are purely attractive, while the two-term potentials possess quasi-hard cores followed by shallow attractive parts. The p–p potentials are less attractive than the n–p potentials due to the presence of repulsive electromagnetic part of the interactions. The quasi-hard core in the two-term potential appears due the second term which dominates over the first term for

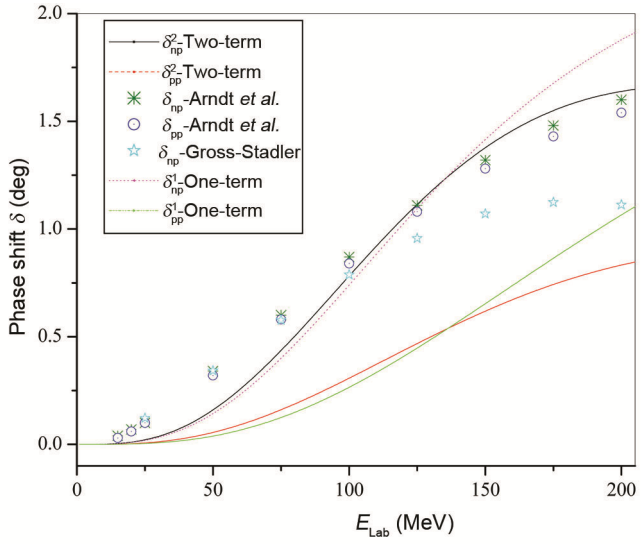

Figure 1. $^3F_2$ phase shifts as a function of laboratory energy.

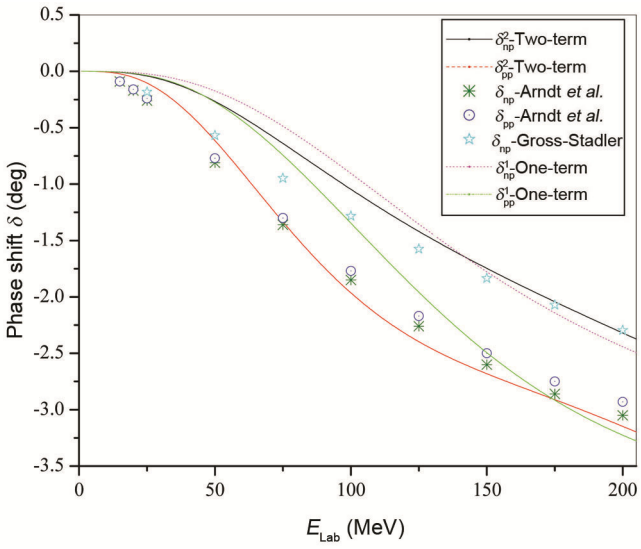

Figure 2. $^3F_3$ phase shifts as a function of laboratory energy.

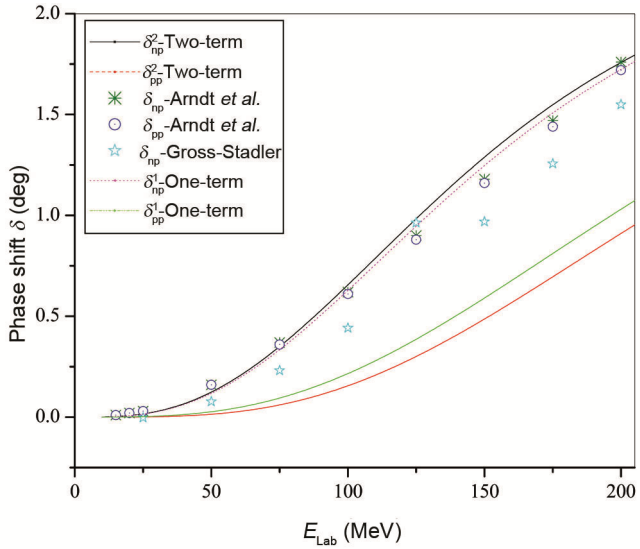

Figure 3. $^3F_4$ phase shifts as a function of laboratory energy.

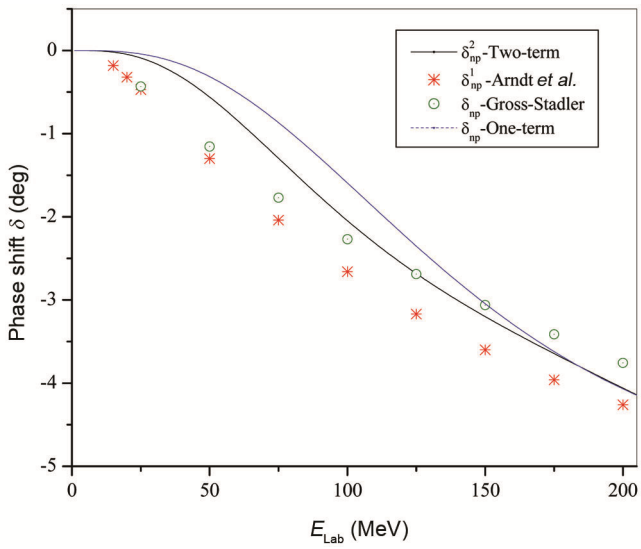

Figure 4. $^1F_3$ phase shifts as a function of laboratory energy.

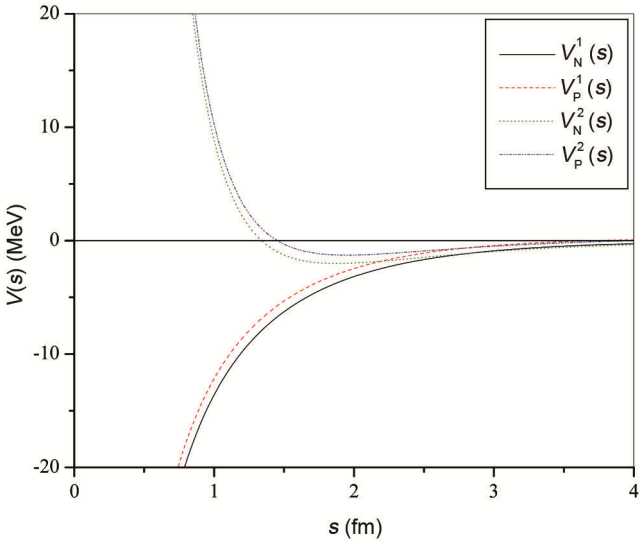

Figure 5. One- and two-term $^3F_2$ potentials.

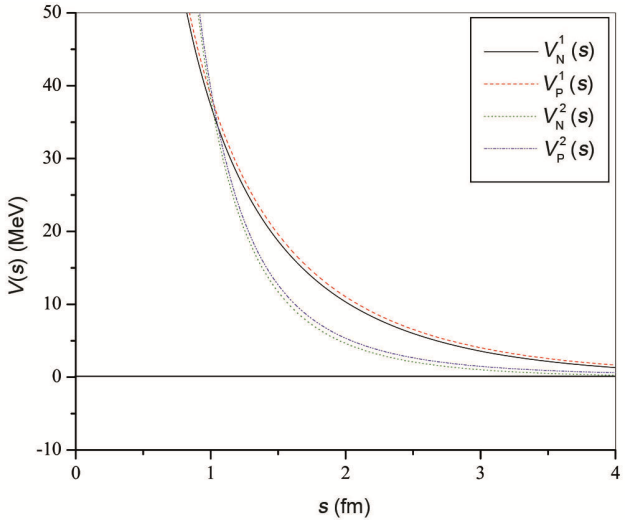

Figure 6. One- and two-term $^3F_3$ potentials.

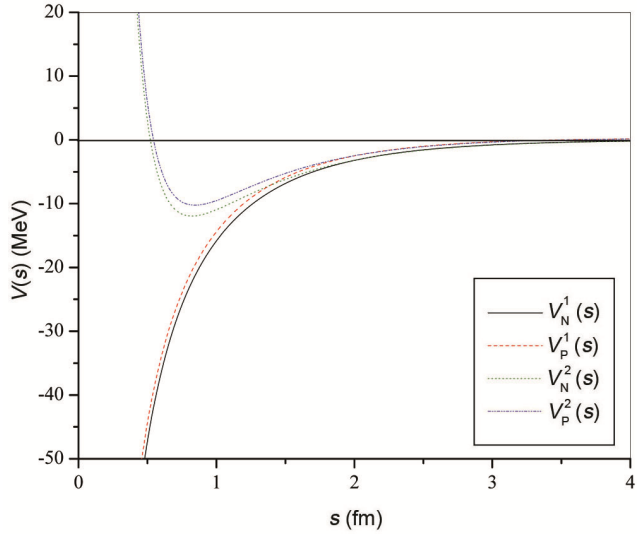

Figure 7. One- and two-term $^3F_4$ potentials.

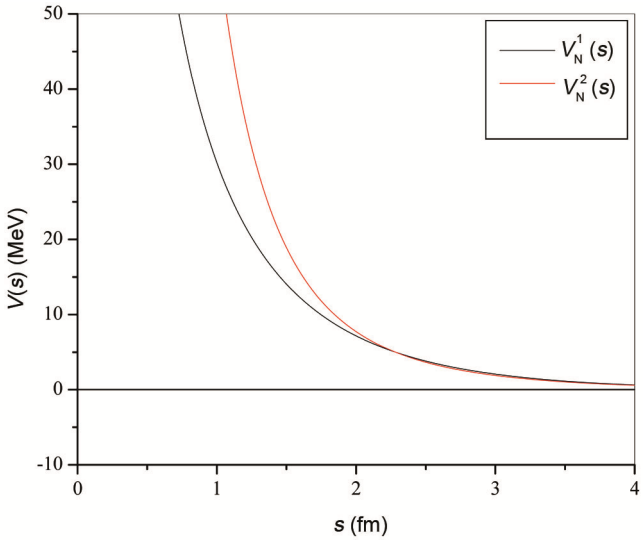

Figure 8. One- and two-term $^1F_3$ potentials.

small values of s. Thus, at small s the resultant potential becomes repulsive in nature. The potentials for $^3F_3$ and $^1F_3$ states shown in Figures 6 and 8 are purely repulsive in nature with strong cores and thus produce negative phase shifts throughout the energy range. This is because the strength parameters S_1 for one-term potential for both the states are negative and as a whole the potential $V_N^1(s)$ becomes positive. On the other hand, the strength parameters S_2 of the two-term potentials for the same states are stronger than S_1 and the resultant potentials become positive. The strong cores in the potentials prevent the particles from overlap.

In general, a full phenomenological potential consists of central, spin–spin, spin–orbit and tensor interactions, and at the same time involves large number of free parameters to reproduce experimental data[18–27]. In contrast, the present model consists of a central part only with four parameters. Our results predict that two-term potential is indispensable for n–n systems rather than the one-term potential. From the foregoing presentation it is clear that the simple Hulthén models, particularly the two-term potential, have the ability to incorporate essential features which are likely to be relevant for nuclear phase shift calculations. Thus, one may conclude by noting that our proposed models will be promising and interesting for further studies on various other systems.

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