A tournament metaphor for dominance hierarchy

Gangan Prathap*

A. P. J. Abdul Kalam Technological University, Thiruvananthapuram 695 016, India

We explore the possibility of using Ramanujacharyulu's tournament metaphor to get new dominance indices. This graph theoretic approach leads to two dimensionless rating measures, a power-weakness ratio (PWR) and a normalized power-weakness difference (nPWD). Three popular dominance indices already in use are the Clutton-Brock *et al.*'s index (CBI), David's score (DS) and frequency-based dominance index (FDI), and we shall show that CBI and FDI are ratio scales like PWR, and DS is an interval scale like nPWD. Unlike the PWR which can become singular (i.e. it ranges from zero to infinity), the nPWD ranges from –1 to +1. The new dominance indices yield unique scores and ranks and can be used interchangeably as they have a monotonic one-to-one relationship.

Keywords: Clutton-Brock *et al.*'s index, David's score, dominance hierarchy, dominance index, frequency-based dominance index, power-weakness ratio, normalized power-weakness difference.

RAMANUJACHARYULU'S paired-comparisons metaphor[1] has been used in the recent past in areas as varied as scientometrics[2,3], evaluating contestants[4], round-robin sports tournaments[5] and multi-criteria decision-making (MCDM)[6]. The power-weakness ratio (PWR), as originally introduced[1] enabled the rating and ranking of players or teams in a tournament or the most influential (powerful) person in a group by simultaneously considering the contestants' 'power' (e.g. strength of players she won against) and her 'weakness' (e.g. weakness of players she lost against). This is exactly the situation that obtains in dominance hierarchy and it will be interesting to see if PWR (a ratio scale) and normalized power-weakness difference (nPWD) (an interval scale) can be used as dominance indices.

Dominance hierarchies are based on paired interaction among individuals so that the most basic interaction is the dyad, where two individuals in a group are paired with each other in a round-robin fashion and the result of their interaction recorded as a win/loss or tie. The hierarchy is then deduced from the dominance matrix that is constructed from all these interactions, in which wins/ties are expressed in relation to each member of the group. Ranking of the individuals is then based on their wins and losses in their dyadic encounters.

*e-mail: gangan_prathap@hotmail.com

In this paper, we shall first introduce the PWR and its derived index, a normalized power-difference index nPWD, and then compare its performance and attributes to three popular indices used for dominance ranking, namely the Clutton-Brock *et al.*'s index (CBI)[7], David's score (DS)[8] and frequency-based dominance index (FDI)[9]. We shall show using dominance matrices from the published literature that the CBI and FDI are ratio scales like PWR and DS is an interval scale like nPWD. Unlike the PWR which is a ratio and can become singular (i.e. it ranges from zero to infinity), the nPWD is interval-based and ranges from –1 to +1. This follows from the fact that each index is a monotonic one-to-one transformation of the other as we shall show later.

Unlike some of the other dominance indices which can lead to tied ranks when there are non-interacting pairs of individuals and reversals, the PWR and nPWD are robust even when there are non-interacting pairs and where there are win-loops, i.e. even when the tournament is incomplete and unbalanced, unique ranks are obtained. This is seen as an attractive feature.

We shall illustrate the concepts of the PWR and the nPWD by starting directly with a dominance matrix[9]. Table 1 is the fictive dominance matrix[9] for each member of hypothetical interactions involving animals a, b, c and d. The power (**A**) and weakness ($\mathbf{A}^{\mathrm{T}}$) forms of the matrix are shown juxtaposed so that in the former, the wins are counted row-wise, and the losses are registered column-wise. The weakness matrix is the transpose of the power matrix; the losses are now counted row-wise and the wins column-wise. Ramanujacharyulu[1] proposed to balance the 'power to influence' through the wins with the 'weakness to be influenced' as conceded through the losses using a measure called the PWR.

If the entries of the power matrix **A** are read row-wise, then for an individual in row i, an entry such as a_{ij} is the actual number of wins by individual i in her interactions with individual j. The matrix can also be read column-wise; now for the individual in column j, the entries a_{ij} are the losses suffered in interactions with i. Thus, row-wise, we see the individual i's 'power to influence' and 'column-wise' we see the individual j's 'weakness to be influenced'. The row-sum corresponding to row i is therefore the sum of wins and this is taken as a measure of the power of individual i.

From the graph theoretical point of view, **A** is the matrix associated with the graph[1]. One interesting property is that it can be raised indefinitely to the kth power, i.e. $\mathbf{A}^{k}$. This is the matrix used to define the 'power of the individual to influence'[1]. So far, the matrix calculations have all proceeded row-wise. For each individual we can find a value $p_i(k)$, which can be called the iterated power of order k of the individual i 'to influence'.

The same operations can be carried out column-wise simply by using the transpose of the matrix, i.e. $\mathbf{A}^{\mathrm{T}}$ and then proceeding row-wise on these transposed elements in the same recursive and iterative manner indicated above[1]. This now defines the 'weakness of the individual to be influenced by'. Again, for each individual we can find a value $w_i(k)$, which can be called the iterated weakness of order k of the individual i 'to be influenced by'.

At this stage we have two vectors of power k – the power vector $p(k)$ and the weakness vector $w(k)$. The elements of the former are the recursive counts of wins and the latter are the recursive counts of losses. Then Ramanujacharyulu's PWR of order k, $r_i(k) = p_i(k)/w_i(k)$ becomes a candidate for a dimensionless measure of dominance. As $k \to \infty$, we get the converged PWR. Usually, reasonable convergence needed for practical rating purposes is obtained after a finite number of steps.

As a ratio scale, the PWR has one weakness; if in a matrix there is an individual that has no losses, then the PWR which is a ratio of wins to losses can become singular. That is, it ranges from zero (sum of wins is zero) to infinity (sum of losses is zero). We can improve on this by introducing a dimensionless interval-based indicator that ranges from –1 (sum of wins are zero) to +1 (sum of losses are zero). This normalized nPWD indicator of order k, is $d_i(k) = p_i(k)/(p_i(k) + w_i(k)) - w_i(k)/(p_i(k) + w_i(k)) = (p_i(k) - w_i(k))/(p_i(k) + w_i(k))$. As $k \to \infty$, we get the converged nPWD. Note that PWR and nPWD are monotonic one-to-one transformations of each other as nPWD = (PWR – 1)/(PWR + 1).

An excellent description and methodology of use of indicators such as the CBI, David's score (DS) and FDI is available in Bang *et al.*[9] and Gammell *et al.*[10]. We shall now highlight only the commonalities and the differences CBI, DS and FDI have with PWR and nPWD. CBI for an individual i is calculated as a ratio of a numerator that registers wins and a denominator that registers losses. The numerator comprises a term B that represents the number of individuals that i has defeated in one or more interactions, and a second term which represents the total number of individuals (excluding i) that those represented in B defeated. In the denominator, there is a term L representing the number of individuals by which i was defeated and a second term that represents the total number of individuals (excluding i) by which those represented in L were defeated. One is added to the numerator and the denominator in the equation because some group members might not have been observed either winning or losing an interaction. Note that CBI counts only individuals and not the actual number of interactions in terms of wins and losses which PWR does. In this sense, PWR has finer granularity than CBI. Also, by taking two terms in the numerator and denominator, it is like the recursive improvement used in PWR but now terminated at the second term (i.e. $k = 2$). In contrast, the PWR is obtained by continuing the recursion iteratively until convergence is achieved. The addition of the unit term in numerator and denominator is to anticipate and avoid the likelihood of a singularity when the denominator term goes to zero. Finally, CBI is a ratio scale just like PWR and should behave in a similar fashion except for the granularity.

FDI[9] can be interpreted as an improvement over CBI by bringing in the granularity that was lost by not counting interactions. It is a ratio scale, and like CBI, is a recursion that terminates at the second iteration. It should therefore have properties similar to CBI and PWR.

Unlike CBI, FDI and PWR, DS[8] is on the interval scale as it is a difference between a count of wins and losses. It is therefore similar to nPWD. However, the recursion is again terminated at the second iteration, unlike the nPWD where we continue the iteration until a reasonable degree of convergence is achieved. And unlike CBI, DS considers all the interactions and not just the individuals and so has finer granularity. Unlike nPWD, DS does not actually count the wins and losses in **A** but works with a matrix of proportions. This is to ensure that DS, which is on an interval scale is dimensionless and is size-independent[2]. Strictly, there is no reason why one should use proportions instead of actual numbers if a dimensionless and size-independent scale like nPWD is used[2]. We shall test these possibilities in the next section.

We now turn to some examples from the published literature to help us compare the PWR and nPWD indicators with the CBI, FDI and DS indices.

The algorithm for Ramanujacharyulu (henceforth Ram's) power and weakness iterations and calculating PWR can be carried out using standard excel spreadsheets and is presented in Prathap *et al.*[2]. Once PWR is found, the monotonic one-to-one transformation to get nPWD is a simple arithmetic exercise. Table 2 shows a comparison of the PWR index with the values for CBI and FDI in Bang *et al.*[9], for the fictive dominance matrix

Table 1. Fictive dominance matrix **A** for each member of hypothetical interactions involving animals a, b, c and d from Bang *et al.*[9]. The power (**A**) and weakness ($\mathbf{A}^{\mathrm{T}}$) forms of the matrix are shown juxtaposed so that in the former, the wins are counted row-wise, and the losses are registered column-wise. The weakness matrix is the transpose of the power matrix

		Loss				Wins
Power		a	b	c	d	P
Win	a	0	1	2	3	6
	b	1	0	2	0	3
	c	0	4	0	1	5
	d	2	0	3	0	5
		Win				Losses
Weakness		a	b	c	d	W
Loss	a	0	1	0	2	3
	b	1	0	4	0	5
	c	2	2	0	3	7
	d	3	0	1	0	4

A given in Table 1. We also show the Pearson's correlation. As expected, the PWR index correlates very well with the FDI as both are indices which are on a ratio scale and have a similar degree of granularity, the only difference being that PWR continues the iteration well beyond two steps and continues until convergence is achieved. The value shown for PWR in Table 2 was reached after 10 steps, i.e. $k = 10$. The correlation between the CBI and FDI indices is also very good, but as expected CBI is unable to yield unique rankings, and ranks b with d. The ranking sequence is different: a > d > c > b for PWR and a > d > b > c for FDI and CBI. Both PWR and FDI give unique rankings except that at the third and fourth ranks there is a reversal of fortune. In such paired-comparison exercises there is no such thing as a ground truth and such results are common in tournament rankings as applied to sports events. It is therefore not possible to say if PWR or FDI actually corresponds more faithfully to reality.

Table 3 compares the nPWD index with the values for DS in Bang *et al.*[9], for the same fictive dominance matrix **A** given in Table 1. Note that DS is actually obtained with a matrix of proportions and we shall repeat this exercise with a matrix of proportions in Table 4 to obtain a new nPWD. Although Pearson's correlation shows that nPWD and DS correlate well as both are indices which are on an interval scale, and both return unique ranks, the ranking sequences are different: a > d > c > b for nPWD but a > d > b > c for DS. At the third and fourth ranks there is a reversal of fortune. The same caveat about ground truth applies here. We shall next carry out the nPWD exercise with a matrix of proportions.

Table 4 show the fictive dominance matrix **A** for each member of hypothetical interactions involving animals a, b, c and d after it is modified to become a matrix of proportions as in Table 2 of Bang *et al.*[9] The power (**A**) and

Table 4. Fictive dominance matrix **A** for each member of hypothetical interactions involving animals a, b, c and d is modified to become a matrix of proportions as in Table 2 of Bang *et al.*[9]. The power (**A**) and weakness ($\mathbf{A}^T$) forms of the matrix are shown juxtaposed so that in the former, the proportion of wins are counted row-wise, and the proportion of losses are registered column-wise. The weakness matrix is the transpose of the power matrix

		Loss				Win
Power		a	b	c	d	P
Win	a	0	0.5	1	0.6	2.1
	b	0.5	0	0.33	0	0.83
	c	0	0.67	0	0.25	0.92
	d	0.4	0	0.75	0	1.15
		Win				Loss
Weakness		a	b	c	d	W
Loss	a	0	0.5	0	0.4	0.9
	b	0.5	0	0.67	0	1.17
	c	1	0.33	0	0.75	2.08
	d	0.6	0	0.25	0	0.85

Table 2. For the fictive dominance matrix[9] **A** given in Table 1, the PWR index is compared with the CBI and FDI indices for the four individuals

Individual	PWR	CBI	FDI
a	2.15	1.43	1.54
b	0.51	1.00	0.94
c	0.63	0.70	0.70
d	1.63	1.00	1.13
Pearson's correlation	PWR	CBI	FDI
PWR	1.00	0.80	0.91
CBI	0.80	1.00	0.98
FDI	0.91	0.98	1.00

Table 3. For the fictive dominance matrix[9] **A** given in Table 1, the nPWD index is compared with the DS index

Individual	DS	nPWD
a	2.30	0.36
b	–0.83	–0.32
c	–2.24	–0.23
d	0.77	0.24
Pearson's correlation	DS	nPWD
DS	1.00	0.90
nPWD	0.90	1.00

Table 5. For the fictive dominance matrix **A** given in Table 4, the nPWD index is compared with the DS index

Individual	DS	nPWD
a	2.30	0.34
b	–0.83	–0.15
c	–2.24	–0.34
d	0.77	0.17
Pearson's correlation	DS	nPWD
DS	1.00	0.99
nPWD	0.99	1.00

Table 6. Matrix containing frequencies of wins and losses in dyadic dominance interactions from de Vries[11]

	a	v	b	h	g	w	e	k	c	y	Wins
a	0	5	4	6	3	0	2	2	3	1	26
v	0	0	0	0	2	1	2	0	7	7	19
b	0	0	0	0	1	1	1	2	2	2	9
h	0	3	0	0	0	0	6	0	2	5	16
g	0	0	0	1	0	2	4	0	3	0	10
w	2	0	0	3	0	0	0	0	2	1	8
e	0	0	0	0	0	0	0	0	0	4	4
k	0	0	0	0	0	0	0	0	2	1	3
c	0	0	0	0	0	1	0	2	0	6	9
y	0	0	0	0	0	0	0	0	2	0	2
Losses	2	8	4	10	6	5	15	6	23	27	106

weakness ($\mathbf{A}^T$) forms of the matrix are again shown juxtaposed so that in the former, the proportion of wins are counted row-wise, and the proportion of losses are registered column-wise. The weakness matrix is the transpose of the power matrix.

Table 7. nPWD and PWR scores for the dominance matrix from Table 6

Individual	nPWD	PWR
a	0.90	19.01
b	0.73	6.37
g	0.64	4.63
v	0.57	3.69
w	0.52	3.16
h	0.37	2.18
k	–0.67	0.20
c	–0.68	0.19
e	–0.78	0.13
y	–0.93	0.04

Table 8. Dominance ranks resulting from Ram's protocol and de Vries' approach[11]

Individual	Ram's rank	de Vries rank
a	1	1
b	2	2
c	8	9
e	9	8
g	3	4
h	6	6
k	7	7
v	4	3
w	5	5
y	10	10
Correlation	Ram's rank	de Vries rank
Ram's rank	1.00	0.98
de Vries rank	0.98	1.00

Table 5 compares the nPWD index with the values for DS in Bang *et al.*[9], for the fictive dominance matrix **A** based on proportions given in Table 4. Note that DS was actually obtained with a matrix of proportions[9] and the nPWD index is now computed for this case. Pearson's correlation shows that nPWD and DS correlate extremely well. Both indices return unique ranks, with identical ranking sequences: a > d > b > c. This leads to a difficult dilemma: whether the use of proportions instead of actual numbers is really justified.

As a second illustrative exercise we take an example from de Vries[11]. Table 6 is the fictive dominance matrix for each member of hypothetical interactions involving 10 individuals. Table 7 shows the scores based on Ram's protocol and Figure 1 is a graphical representation of the arrangement of the ranks of the 10 individuals. Note that all 10 have unique scores. The monotonic one-to-one mapping from PWR to nPWR is clearly evident. In Table 8 we compare Ram's ranks and the ranks obtained by de Vries[11]. The rank correlation is seen to be excellent. Once again it is clear that there is no such thing as a ground truth in such comparisons. From the point of view of naturalness, Ram's procedure is simple, treats wins and losses which are asymmetrical in a symmetric fashion by the separate row-wise and column-wise handling

Table 9. Matrix showing dominance interactions among cockroaches (from Bell and Gorton[13])

	D	A	E	C	B	Wins	CDR
D	0	9	12	6	27	54	1
A	10	0	9	12	12	43	1.2
E	2	5	0	0	2	9	2.1
C	3	3	0	0	2	8	2.5
B	2	3	0	4	0	9	3
Losses	17	20	21	22	43	123	

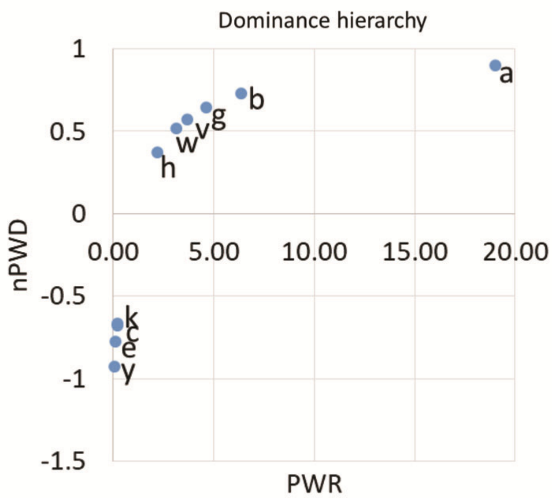

Figure 1. Arrangement of dominance ranks for the 10 individuals based on PWR and nPWD scores from Table 7.

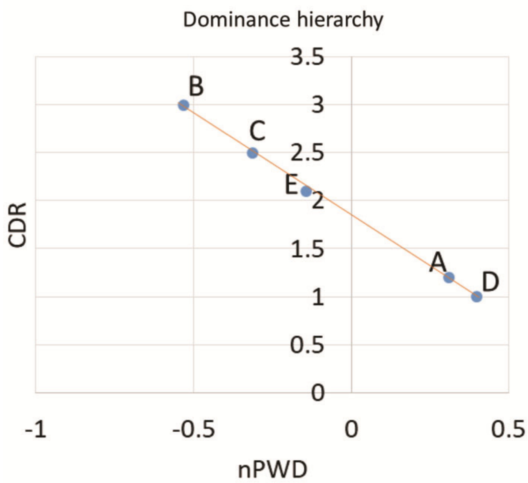

Figure 2. Arrangement of dominance scores for the five individuals based on PWR and CDR scores from Table 10.

Table 10. nPWD and CDR scores for the dominance matrix from Table 9

Individual	nPWD	CDR
D	0.40	1.00
A	0.31	1.20
E	–0.14	2.10
C	–0.31	2.50
B	–0.53	3.00
Pearson's correlation	nPWD	CDR
nPWD	1.00	–1.00
CDR	–1.00	1.00

of the eigenvalue solution[1] and then uses the power and weakness scores to derive either scale-based or interval-based indices which are unique.

For another interesting illustrative example we consider Boyd and Silk's[12] method of assigning cardinal dominance ranks (CDR) with nine data sets of dominance interactions among five captive male cockroaches, *Nauphoeta cinerea*, originally reported by Bell and Gorton[13]. Table 9 displays the matrix showing dominance interactions for the third data set among cockroaches with the the animals ranked according to their cardinal dominance rank. Table 10 gives the nPWD from Ram's protocol and CDR scores for the dominance matrix from Table 9. Table 10 also shows the remarkable negative correlation between nPWD and CDR scores and this is graphically represented in Figure 2. Again, all five individuals have unique scores. CDR, which is rarely used as a dominance measure is apparently an interval scale measure like nPWD, with the ranking score inversely related to the nPWD score.

Ramanujacharyulu's tournament metaphor gives us two new dominance indices. Both are dimensionless rating measures. The PWR is an indicator on a ratio scale like CBI and the FDI. The nPWD is an indicator on an interval scale and is similar to DS. Both PWR and nPWD seem to yield unique ranks.

From the point of view of naturalness, which is a desirable property of mathematical models, Ram's procedure is simple and direct. It treats wins and losses which are asymmetrical in a symmetric fashion by the separate row-wise and column-wise handling of the eigenvalue solution[1] and then uses the power and weakness scores to derive either scale based or interval based indices which are unique. No assumptions need be made about a linear or near-linear hierarchy or about the probabilities of winning or losing[11].

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Molecular association and gelling characteristics of curdlan

Jagadeeshwar Kodavaty[1,*], Gayathri Venkat[2], Abhijit P. Deshpande[3] and Satyanarayana N. Gummadi[2]

[1]Department of Chemical Engineering, University of Petroleum and Energy Studies, Bidholi via-Prem Nagar, Dehradun 248 007, India
[2]Department of Biotechnology, BJM School of Biosciences, Indian Institute of Technology Madras, Chennai 600 036, India
[3]Department of Chemical Engineering, Indian Institute of Technology Madras, Chennai 600 036, India

Curdlan, a polysaccharide known to be used in the food packaging industry that gels out when exposed to higher temperatures, leading to different kinds of gel structures. The triple-helical structural formations of the

*For correspondence. (e-mail: j.kodavaty@ddn.upes.ac.in)