System of kinematical conservation laws

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In a wide range of physical phenomena, we find surfaces Ω_t evolving in time t, which need mathematical treatment. Here, we briefly review the theory of a system of conservation laws known as the kinematical conservation laws (KCLs), which govern the evolution of these surfaces. KCLs are the most general equations in conservation form which govern the evolution of Ω_t with physically realistic singularities. A special type of singularity is a kink, which is a point on Ω_t when it is a curve in two dimensions and a curve on Ω_t when it is a surface in three dimensions. Across a kink, the normal direction $\boldsymbol{n}$ to Ω_t and the normal velocity m of Ω_t are discontinuous. This article is aimed at non-experts in the field. Readers may refer to the literature for more details.

Keywords: Curves and surfaces, kinematical conservation laws, kink, ray theory.

THE movement of a surface Ω_t is determined by the movement of its points according to a law depending on the medium in which Ω_t evolves. The path of a point on Ω_t is called a ray. In this article, for simplicity, we shall consider only an isotropic motion of Ω_t which indicates that the ray velocity $\boldsymbol{\chi}$ is in the direction of the normal $\boldsymbol{n}$, i.e. $\boldsymbol{\chi} = m\boldsymbol{n}$, where m is an appropriately defines the velocity of Ω_t.

Consider Ω_t in two-dimensions (2D) with a ray making an angle θ with the x-axis. Then the ray velocity is $m\boldsymbol{n} = m(\cos\theta, \sin\theta)$ and we can track the successive positions of Ω_t by solving the ray direction equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = m\cos\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = m\sin\theta, \tag{1}$$

and the diffraction equation (the derivation requires the eikonal equation, which is a first-order PDE)

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \left(-\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\right)m. \tag{2}$$

The operator on m on the right-hand side of eq. (2) represents the space rate of change along Ω_t, showing that θ changes along the ray due to a gradient of m along Ω_t.

When m is constant on Ω_t, the rays are straight lines (as in the case of a wavefront in gas with uniform and constant properties) and Ω_t may develop a caustic (Figure 1 left). However, when the rays have built in them, the effect of genuine nonlinearity (Genuine nonlinearity comes into play when m in the ray velocity depends on the amplitude of the wavefront. In this case, m is no longer constant, the rays stretch (implied by eq. (1)) and diffract (implied by eq. (2)). The concept of genuine nonlinearity and its effect on the distortion of the shape of a wave have been explained in two popular articles using the language of physics and only a few mathematical equations[12,13].) of the equations governing a medium in which Ω_t propagates, a new type of singularity, called kink appears on Ω_t (ref. 14) (Figure 1, right, where two kinks appear and are shown by dots). Both the ray theory and the level set theory to solve the eikonal equation are inadequate to study the formation and evolution of a kink.

The distinguishing feature of a kink is the appearance of a discontinuity in the normal direction $\boldsymbol{n}$ and the velocity m of Ω_t across the kink. Examples of Ω_t with discontinuities in $\boldsymbol{n}$ and m across a 2D surface on Ω_t are plenty. They were observed in experimental results[18] and we aimed to capture these results mathematically. When discontinuities in $\boldsymbol{n}$ and m appear on Ω_t, the governing PDEs of Ω_t, i.e. the eikonal equation and ray theory breakdown, then we need to go to the more basic formulation for the evolution of a curve or surface, which is kinematical conservation law (KCL). Conservation laws are formulated in terms of integrals, which remain valid even if singularities (like discontinuities in functions) appear on Ω_t. Two-dimensional KCL for the evolution of a curve was formulated in 1992 by K. W. Morton, the present author (P.P.) and Renuka Ravindran, and has been extensively used to determine new properties of weakly nonlinear wavefronts and shock fronts. The results have been published in several important journals. Three-dimensional KCL was first formulated in 1994 by Mike Giles, P.P. and Renuka Ravindran, but it was completed much later by K. R. Arun and P.P. Papers on 3D KCL and its applications have been published since 2009 (see ref. 17 for details). We wish to emphasize that all these interesting and physically realistic results were obtained without clearly identifying the density of the conserved variable and flux for KCL. In the present article, we shall identify these and complete the KCL theory.

Readers not familiar with the theory of conservation laws may consult the literature[12,13]. Here we give an example of a conservation law.

Conservation of mass in fluid mechanics

Let us consider fluid flow in a one-dimensional pipe along the x-axis and two sections at the x_ℓ and x_r. Let $\rho(x, t)$

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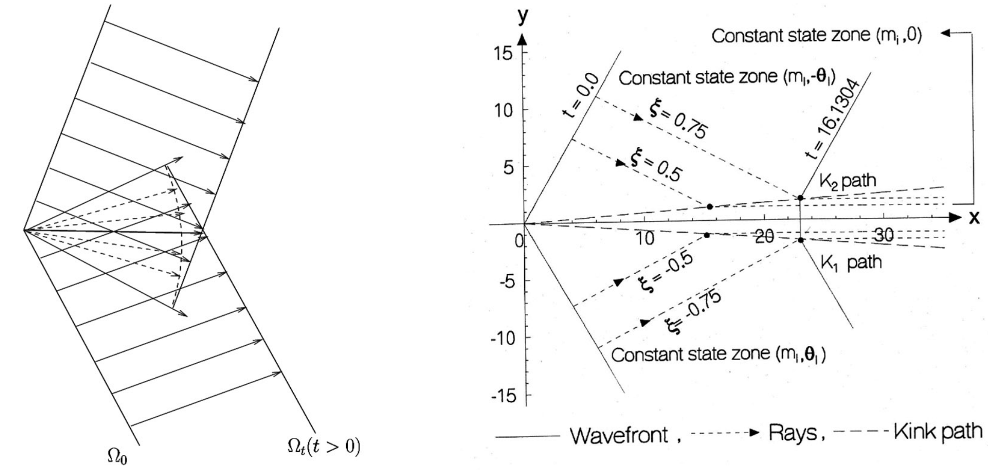

Figure 1. (Left) Representing the linear theory which starts with a concave wedged shape front. (Right) The caustic of the linear theory is completely resolved by the presence of genuine nonlinearity present in kinematical conservation law (KCL) based weakly nonlinear ray theory. This is an exact solution. We find that two kinks have appeared. Across a kink directions of the tangents of the wavefront and directions of the rays suffer jumps. These features can be captured only by a KCL-based theory.

List of some frequently used symbols and abbreviations

$\boldsymbol{x}$	: $(x_1, x_2, ..., x_d)$ a point $\mathbb{R}^d$, d-dimensional physical space or d-D.
Ω_t	: It is the position of a moving surface at time t in $\boldsymbol{x}$-space (we consider only $d = 2, 3$) (the results can be extended easily to a surface in space of arbitrary dimensions[16,17]).
KCL	: A system of kinematical conservation laws which govern the evolution Ω_t.
d-D KCL	: KCL in d-dimensional $\boldsymbol{x}$-space.
WNLRT	: Weakly nonlinear ray theory.

represent the mass density and $u(x, t)$ the fluid velocity (u can be positive or negative) as a function of space and time. The total fluid contained between the two sections at time t is $\int_{x_\ell}^{x_r} \rho(\xi,t)\mathrm{d}\xi$. The flux (per unit time) of mass at x_ℓ is $\rho(x_\ell, t)u(x_\ell, t)$ and that at x_r is $\rho(x_r, t)u(x_r, t)$. When a fluid is not created or annihilated (for example, by a chemical reaction), a balance between the time rate of change of mass in these two sections and the fluxes at the two ends is expressed by the conservation law of mass of the fluid, which states that the rate of change of a fluid contained in any section is due to the flux through its ends. In mathematical terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_\ell}^{x_r} \rho(x, t)\mathrm{d}x = \rho(x_\ell, t)u(x_\ell, t) - \rho(x_r, t)u(x_r, t). \qquad (3)$$

When the mass density and fluid velocity are differentiable, we can deduce from this

$$\rho_t + (\rho u)_x = 0, \qquad (4)$$

which gives the Euler's equation of continuity

$$\rho_t + u\rho_x + \rho u_x = 0. \qquad (5)$$

Remark 1.1. An important convention in the theory of conservation laws: equation (4) is assumed to represent the integral form, i.e. eq. (3) of the conservation law. This is only for the simplicity of notation.

Ray coordinates (ξ, t) for an isotropic motion of a moving curve Ω_t in the (x, y)-plane and 2D KCL

Let the curve Ω_t be represented by

$$\Omega_t : \varphi(x, y, t) = 0. \qquad (6)$$

We write a parametric equation of Ω_t: $\varphi(x, y, t) = 0$ in the form

$$\Omega_t: x = x(\xi, t), \ \ y = y(\xi, t), \qquad (7)$$

where ξ is the parametric variable on Ω_t at time t.

Ray coordinates

(ξ, t) forms ray coordinates in the (x, y)-plane such that in eq. (7), the constant values of t give the positions of the propagating curve Ω_t at different times and ξ = constant represents a ray. The results below are valid in a domain $\mathcal{D}_p$, swept by Ω_t, of the (x, y)-plane. Let g be the metric associated with the coordinate ξ, i.e. $g\,\mathrm{d}\xi$ is an element of the distance along Ω_t. Note that if m is the velocity of a point on the ray, then $m\,\mathrm{d}t$ is an element of the distance along a ray (Figure 2). Therefore, m is the metric associated with t.

In the ray coordinates (ξ, t), the time rate of change along a ray, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + m(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y})$ becomes $\frac{\partial}{\partial t}$ and the space rate of change along Ω_t appearing on the right-hand side in eq. (2) becomes $\frac{\partial}{g\partial\xi}$. We can also show that $g_t = m\theta_\xi$. Finally in the ray coordinates (ξ, t), eqs (1) and (2) (along with an equation for g) become

$$x_t = m\cos\theta,\ y_t = m\sin\theta. \tag{8}$$

$$g_t = m\theta_\xi,\ \theta_t = -\frac{1}{g}m_\xi. \tag{9}$$

The results contained in eq. (9) are physically realistic. We verify it for an expanding circular curve Ω_t with a centre at the origin and with a constant velocity $m > 0$ at a time, say at $t = t_0$. Then it remains circular all the time. Since $m_\xi = 0$, eq. (9) (the second equation) implies that θ remains constant along a ray, which is a straight line. Since $\theta_\xi > 0$, eq. (9) (the first equation) shows that g increases with t. This is consistent with the fact that the arc length $\int_{\xi_\ell}^{\xi_r} g\mathrm{d}\xi$ of Ω_t keeps increasing in the fixed interval (ξ_ℓ, ξ_r).

We shall give the formulation of 2D KCL in terms of the density of 'KCL conserved variable' and 'KCL flux'. Kinematics refers to the study of the motion of bodies without reference to mass or force. KCL involves purely geometric objects and is in the (ξ, t)-plane.

Proposal for density of KCL conserved variable

Associated with Ω_t, there are two important vectors.

(i) Ray velocity $(m\cos\theta, m\sin\theta)$ obtained by the normal direction $\boldsymbol{n}$ of Ω_t (see eq. (1)) multiplied by the metric m.

Ray coordinate system associated with Ω_t: (ξ, t)

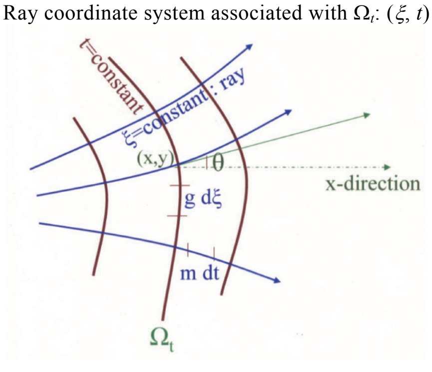

Figure 2. In the (x, y)-plane, ξ = constant is a ray and t = constant is Ω_t. (x, y) is a point on the moving curve Ω_t at time t. θ is the angle which the ray at (x, y) makes with the x-direction.

(ii) A vector $(-g\sin\theta, g\cos\theta)$ tangential to the curve Ω_t. This is obtained by multiplying the metric g by the unit vector $(-\sin\theta, \cos\theta)$ along Ω_t (see eq. (2)).

We propose that the conserved variable in KCL is the tangential vector mentioned in (ii) and the flux vector is the ray velocity mentioned in (i).

The balance between time rate of change of total conserved quantity (now a tangential vector with two components on Ω_t) from the point identified by ξ_ℓ on Ω_t to that identified by ξ_r on Ω_t, and the flux from the two ends is expressed as a conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\xi_l}^{\xi_r}(-g\sin\theta, g\cos\theta)(\xi, t)\mathrm{d}\xi = \{(m\cos\theta, m\sin\theta)\}(\xi_\ell, t) - \{(m\cos\theta, m\sin\theta)\}(\xi_r, t). \tag{10}$$

If θ is a constant on Ω_t, then the front is a straight line at time t. If m is a constant on it, then every point at time t moves with the same velocity. When both are constant on Ω_t, the curve propagates as a straight line parallel to itself and eq. (10) shows that $\int_{\xi_\ell}^{\xi_r}(-g\sin\theta, g\cos\theta)(\xi, t)\mathrm{d}\xi$ on Ω_t between the points corresponding to ξ_ℓ and ξ_r remains constant as time evolves, i.e. it remains conserved.

Symbolically, eq. (10) is denoted by (see Remark 1.1)

$$(g\sin\theta)_t + (m\cos\theta)_\xi,\ (g\cos\theta)_t - (m\sin\theta)_\xi = 0. \tag{11}$$

We wrote the form of KCL in our earlier article (by Morton, Prasad and Ravindran) without mentioning the density vector of the conserved quantity and the flux vector. The differential form of KCL eq. (11) is eq. (9), which has been derived from the ray equations.

Now we state an important theorem.

Theorem: KCL and ray equations are equivalent for the evolution of a smooth Ω_t. Full proof this theorem includes:

(1) KCL implies the ray equations.
(2) The ray equations imply KCL.

A proof is available in ref. 15 (theorem 4, pp. 14–15).

Three-dimensional system of KCL

Unlike 2D KCL, the theory of 3D KCL is quite involved. Here, we describe the results briefly; the details of its theory and applications are available in the literature[1–4,16].

Ray equations in 3D space

We denote the components of the unit normal to Ω_t as

$$\boldsymbol{n} = (n_1, n_2, n_3),\ |\boldsymbol{n}| = 1. \tag{12}$$

For an isotropic evolution of Ω_t the ray equations take simple forms

$$\frac{\mathrm{d}x}{\mathrm{d}t} = m\boldsymbol{n}, \tag{13}$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\boldsymbol{L}m := -(\nabla - \boldsymbol{n}\langle \boldsymbol{n}, \nabla\rangle)m. \tag{14}$$

The operator $\boldsymbol{L}$ defined above is obtained by subtracting from the gradient ∇ its component in the normal direction $\boldsymbol{n}$; hence it represents a tangential derivative on Ω_t.

Ray coordinate system

Let the surface Ω_t: $\varphi(x_1, x_2, x_3, t) = 0$ at any time t be represented by

$$\Omega_t: \boldsymbol{x} = \boldsymbol{x}(\xi_1, \xi_2, t), \tag{15}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$ is a set of surface coordinates which also evolve with time t (Figure 3). At a fixed time t, the surface Ω_t in $\boldsymbol{x}$-space is generated by a two-parameter family of curves, the parameters being ξ_1 and ξ_2. Along a member of the first family of curves, ξ_1 varies and ξ_2 is a constant. Similarly, along a member of the second family of curves, ξ_2 varies and ξ_1 is a constant. Through each point $\boldsymbol{\xi}$ on Ω_t a ray passes in the normal direction $\boldsymbol{n}$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be unit vectors along the ξ_1 and ξ_2 families of the coordinates (Figure 3).

Density of the conserved vector and flux vectors for 3D KCL

Let the metric associated with the surface coordinate ξ_p be g_p, $p = 1, 2$. Then $g_p\mathrm{d}\xi_p$ (no sum over the repeated subscript here) is an element of distance along the coordinate line through which ξ_p varies. The speed of a point moving along the ray with velocity is m then, while moving with the ray velocity, $m\mathrm{d}t$ is the displacement along the ray in time $\mathrm{d}t$. Thus m is the metric associated with the coordinate t.

As in the case of the 2D KCL, we take the density of the conserved vector along the family of coordinates to be $g_1\boldsymbol{u}$ and that along the second family of coordinates to be $g_2\boldsymbol{v}$. We also propose the flux vectors along both families to be the ray velocity $m\boldsymbol{n}$.

We first consider the formulation in the subspace of the (x_1, x_2, x_3)-space in which ξ_2 is a constant. This means we consider the formulation in the (ξ_1, t)-plane of (ξ_1, ξ_2, t)-space. The conserved density vector is $g_1\boldsymbol{u}$ and flux is $m\boldsymbol{n}$. The integral formulation of the first set of conservation laws is:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\xi_{1\ell}}^{\xi_{1r}} g_1(\xi_1, \xi_2, t)\boldsymbol{u}_1(\xi_1, \xi_2, t)\mathrm{d}\xi_1\\ &= m(\xi_{1\ell}, \xi_2, t)\boldsymbol{n}(\xi_{1\ell}, \xi_2, t) - m(\xi_{1r}, \xi_2, t)\boldsymbol{n}(\xi_{1r}, \xi_2, t),\\ &\xi_2 = \text{constant}.\end{aligned} \tag{16}$$

Then we consider the formulation in the (ξ_2, t)-plane of the (ξ_1, ξ_2, t)-space. The conserved density vector is $g_2\boldsymbol{u}$ and flux is $m\boldsymbol{n}$. Now we write the second set of conservation laws as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\xi_{2\ell}}^{\xi_{2r}} g_2(\xi_1, \xi_2, t)\boldsymbol{u}_2(\xi_1, \xi_2, t)\mathrm{d}\xi_2\\ &= m(\xi_1, \xi_{2\ell}, t)\boldsymbol{n}(\xi_1, \xi_{2\ell}, t) - m(\xi_1, \xi_{2r}, t)\boldsymbol{n}(\xi_1, \xi_{2r}, t),\\ &\xi_1 = \text{constant}.\end{aligned} \tag{17}$$

The symbolic form of the conservation laws, viz. eqs (16) and (17) are

$$(g_1\boldsymbol{u})_t - (m\boldsymbol{n})_{\xi_1} = 0,\ (g_2\boldsymbol{v})_t - (m\boldsymbol{n})_{\xi_2} = 0. \tag{18}$$

$\boldsymbol{u}$ and $\boldsymbol{v}$ have three components each and hence the 3D KCL has six equations.

Theorem 3.1. 3D KCL eq. (18) and the ray equations (eqs (13) and (14)) are equivalent for the evolution of a smooth Ω_t.

We stated this theorem for 2D KCL earlier in the article. Here too, we omit the proof since different proofs are available in the literature.

An explicit differential form of 3D KCL is available in ref. 1.

Geometrical solenoidal constraint

This is an interesting aspect of KCL in three and higher dimensions. Apart from 3D KCL eq. (18), we can easily

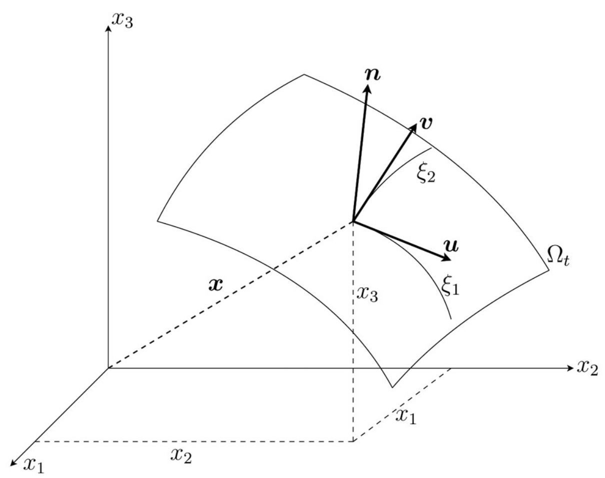

Figure 3. A ray coordinate system of a surface Ω_t.

derive three more scalar equations contained in the vector equation

$$(g_2\boldsymbol{v})_{\xi_1} - (g_1\boldsymbol{u})_{\xi_2} = 0. \tag{19}$$

Equation (19) gives purely geometrical results, as m does not appear in it and we call it a geometrical solenoidal constraint. From eq. (18) we can show that $(g_2\boldsymbol{v})_{\xi_1} - (g_1\boldsymbol{u})_{\xi_2}$ does not depend on t. Hence, for any choice of coordinates ξ_1 and ξ_2 on Ω_0, eq. (19) is satisfied at $t = 0$, then it follows that the equation is satisfied on Ω_t for all $t > 0$.

Control of Jordan mode by geometrical solenoidal constraint

We now mention some applications of KCL. If we consider KCL-based weakly nonlinear ray theory (WNRT) or shock ray theory (SRT) (see refs 6 and 14, and any of the references mentioned above), the 2D problems remain simple but 3D problems become complex. For example, the 3D WNLRT equation becomes degenerate. It has an eigenvalue 0 with a multiplicity 5, but the number of eigenvectors is only 4. In such a case, unwanted Jordan mode generally appears in a numerical solution, which grows in time (for more details, see ref. 2).

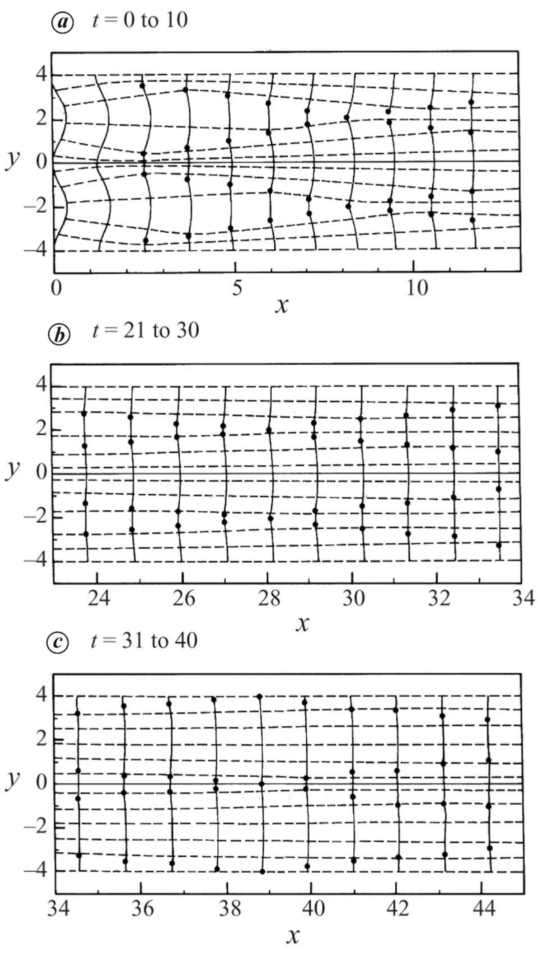

Figure 4. Successive positions of an initially sinusoidal shock front (shown by continuous line) plotted at $t = 0, 1, 2, 3, \ldots, 40$ and rays (shown by broken lines). Kinks appear after $t = 1$ and are shown by dots. In the figure we see a kink, $\mathcal{K}_3$, from a lower period moving upwards and another kink, say $\mathcal{K}_1$, from the upper period moving downwards resulting in interactions $\mathcal{K}_3\mathcal{K}_1 \rightarrow \mathcal{K}_1\mathcal{K}_3$ occurring many times. The shock has become almost straight and rays are parallel to the x-axis from $t = 31$ to 40 (reproduced from ref. 10).

KCL-based WNLRT and SRT

KCL contain a set of under-determined equations. For example, the 2D KCL equations are two in number for three dependent variables m, θ and g. The 3D KCL equations are six in number for seven dependent variables (two independent components of $\boldsymbol{u}$, two independent components of $\boldsymbol{v}$, g_1, g_2 and m). When the moving curves represent a physically observable curve such as a wavefront or a shock front or the crest line of a curved solitary wave on the surface of water[5], we hope to close the KCL system of equations. We do not go into details, but reproduce figures from some references mentioned above.

Exact solutions showing resolution of a caustic in 2D by WNLRT

Figure 1 (right) is an exact solution and has been reproduced from ref. 14. The results are briefly described in the figure caption and in the introduction of this article.

Evolution of a shock front of periodic shape in two space dimensions

For a shock front, we denote the normal velocity of the shock by M (this means we replace m with M). We take the initial shock front Ω_0 to be in a periodic sinusoidal shape

$$x = 0.2 - 0.2\cos\left(\frac{\pi y}{2}\right). \tag{20}$$

We choose a smooth initial shock Mach number (appropriately non-dimensionalized shock speed) to be uniform and as $M_0 = 1.2$. Figure 4 (reproduced from ref. 10) gives the shape of the shock at various times. The dots represent kinks which appear later. Such a long tracking of a shock front with kinks is difficult by any method other than a KCL-based formulation.

Evolution of a shock front of periodic shape in three space dimensions (reproduced from ref. 3)

We consider here the initial shock front Ω_0 to be of a periodic shape in x_1- and x_2-directions.

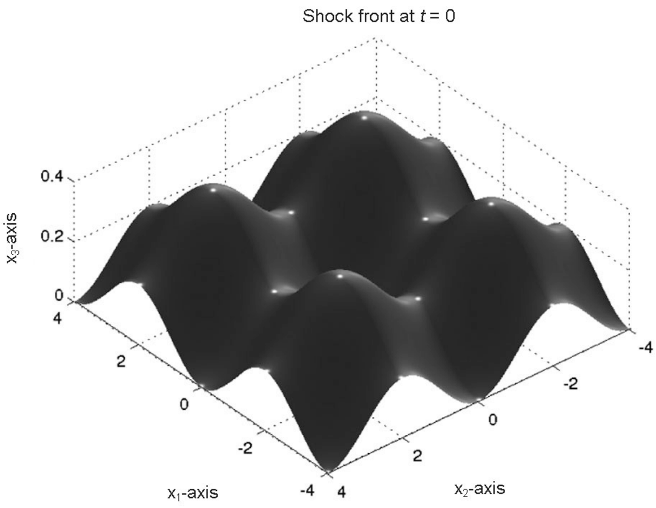

Figure 5. Initial shock front in the shape of a smooth periodic pulse.

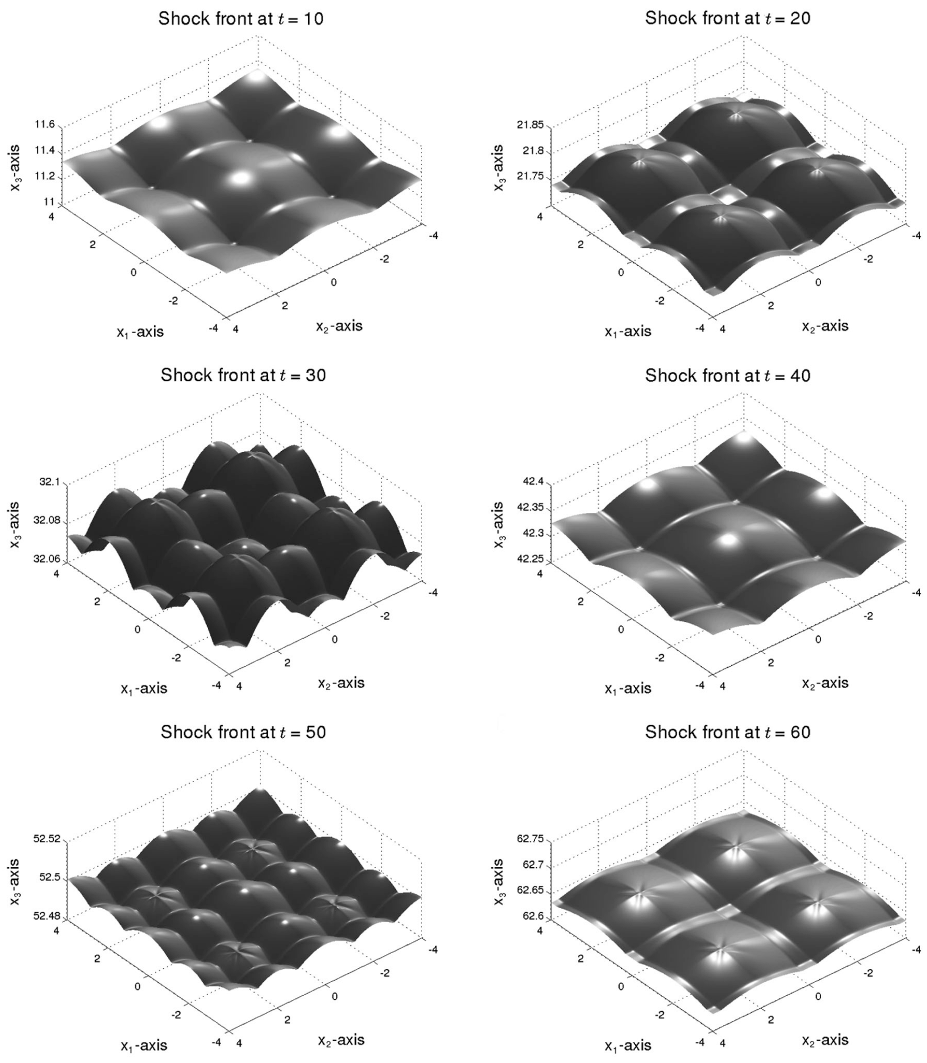

Figure 6. Shock front Ω_t starting initially in a periodic shape with $M_0 = 1.2$. It develops a complex pattern of kinks and ultimately becomes planar.

$$\Omega_0 : x_3 = k\left(2 - \cos\left(\frac{\pi x_1}{a}\right) - \cos\left(\frac{\pi x_2}{b}\right)\right), \tag{21}$$

with the constants $\kappa = 0.1$, $a = b = 2$. In Figure 5, we give the plot of the initial shock front Ω_0, which is a smooth pulse without any kink lines. We choose the Mach number to be uniform and equal to 1.2 on Ω_0. Though the initial shock front is smooth, a number of kink lines appear in each period as time evolves.

Now we describe an interesting process of interaction of these kink lines using a number of plots of the shock front at different instances. In Figure 6, we give the surface plots of the shock front Ω_t at times $t = 10, 20, 30, 40, 50, 60$ in two periods in each of x_1- and x_2-directions. As mentioned above, the initial shock front is smooth, with no kink lines. The front Ω_t moves up in the x_3-direction and develops several kink lines. Four kink lines parallel to the x_1-axis and four parallel to the x_2-axis can be seen in the figure on the shock front at times $t \geq 10$. These kink lines are formed before $t = 10$, say about $t = 2$. The amplitude at $t = 0$ is 0.2 and at $t = 60$, it is less than 0.05; thus, the shock front at $t = 60$ has almost become planar.

Conclusion

KCL is a purely geometric system of conservation laws that govern the evolution of a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$. The curve or surface, as the case may be, can admit singularities which are captured by the weak solutions of KCL. This article provides an exposition on the density of conserved variables of KCL, which involves the curved distances along the moving curve or surface.

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