Gravity, Bose–Einstein condensates and Gross–Pitaevskii equation

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We explore the effect of mutual gravitational interaction between ultra-cold gas atoms on the dynamics of Bose–Einstein condensates (BEC). Small-amplitude oscillation of BEC is studied by applying variational technique to reduce the Gross–Pitaevskii equation, with gravity included, to the equation of motion of a particle moving in a potential. According to our analysis, if the *s*-wave scattering length can be tuned to zero using Feshbach resonance for future BEC with occupation numbers as high as $\approx 10^{20}$, there exists a critical ground state occupation number above which the BEC is unstable, provided that its constituents interact with a $1/r^3$ gravity at short scales.

Keywords: Bose–Einstein condensate, Gross–Pitaevskii equation, instability, large extra dimension gravity.

Introduction

GRAVITY is the weakest of all forces. This is essentially due to the smallness of Newton's gravitational constant (or, equivalently, largeness of Planck mass), measured on scales larger than tens of kilometres[1]. However, to resolve issues pertaining to naturalness and hierarchy problems in the Standard Model of particle physics, it has been conjectured that if large extra dimensions exist, the effective gravitational coupling strength can be larger at sub-millimetre scales[2,3]. With the advent of exciting precision experiments involving Bose–Einstein condensation of alkali atoms and molecules at ultra-low temperatures[4,5], it is but natural to study effects of enhanced gravity ensuing from large extra dimensions (LED) on such macroscopic quantum phenomena.

In this context, Dimopoulos and Geraci have proposed an interesting experiment to probe gravity at sub-micron scale through measurements of relative phase evolution rates in Bose–Einstein condensates (BEC) prepared in coherent superposition of states localized at two distinct potential wells, both situated near a moving wall of alternating gold and silver metal objects that form a periodic massive source of gravity[6]. Similarly, Sigurdsson has suggested measuring fringe shifts of an interfering pair of BEC falling past a long and narrow cylindrical mass in order to estimate modified transverse gravitational acceleration, provided that the LED sub-millimetre scale is in excess of 0.01 mm (ref. 7).

Interestingly enough, the typical separation between atoms in ultra-cold gases is only about a few hundred nanometres. This induces one to explore effects of mutual gravitational interaction between individual atoms of a BEC on its quantum dynamics, and ask whether such weak but long-range forces can lead to instabilities. In this article, we carefully examine some aspects of these ideas using variational method.

Gross–Pitaevskii equation and large extra dimensions-induced gravity

For N identical bosons constituting a dilute BEC at temperature $T \approx 0$ K, the many-body wavefunction $\Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$ describing the condensate can be expressed up to a good approximation (assuming that the bosons interact weakly with each other) as

$$\Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) \cong \prod_{j=1}^{N} \psi(\vec{r}_j),$$

where $\psi(\vec{r})$ is the normalized ground state wavefunction for a single boson. As each boson, in this case, is approximately in the same state, $\psi(\vec{r})$ acts as the condensate wavefunction.

In the $T = 0$ K mean field approximation, dynamical evolution of the condensate wavefunction $\psi(\vec{r}, t)$ (normalized to unity) is, to a good extent, governed by the Gross–Pitaevskii equation (GPE)

$$i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} + N \int V(\vec{r} - \vec{u}) |\psi(\vec{u}, t)|^2 \mathrm{d}^3 u \right] \psi(\vec{r}, t), \quad (1)$$

where m, $V_{\text{ext}}(\vec{r})$ and $V(\vec{r})$ are the boson mass, the trap potential energy required to confine the BEC and the interaction potential energy between two bosons respectively.

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For the present purpose, the interaction potential energy V in eq. (1) is a combination of s-wave scattering potential and the inter-bosonic gravitational potential energy V_g, so that

$$V(\vec{r}-\vec{u}) = \frac{4\pi\hbar^2 a}{m}\delta^3(\vec{r}-\vec{u}) + V_g(|\vec{r}-\vec{u}|), \tag{2}$$

where a is the s-wave scattering length.

Substitution of eq. (2) in eq. (1) results in the standard GPE[5]

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} + Ng|\psi(\vec{r},t)|^2 + N\int V_g(|\vec{r}-\vec{u}|)\,|\psi(\vec{u},t)|^2 \mathrm{d}^3u\right]\psi(\vec{r},t), \tag{3}$$

where $g \equiv (4\pi\hbar^2 a/m)$.

It is interesting to note that the quantum dynamics of a BEC, comprised of ultra-cold bosonic atoms anchored to a planar honeycomb optical lattice and interacting weakly with one another via a contact interaction much like the first term of the RHS of eq. (2), is described by a nonlinear Dirac equation[8]. Furthermore, the pseudospin degrees of freedom associated in this case with the two inequivalent sites of the sublattice display half integral spin angular momentum features, stretching the graphene analogy farther, even though the system is a bosonic one[9].

The GPE of eq. (3) can be easily derived from the following action by demanding it to be stationary under infinitesimal variations of ψ and ψ^*

$$S = \int \mathrm{d}t \int \mathrm{d}^3r\mathcal{L}, \tag{4}$$

where the Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{i\hbar}{2}\left\{\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right\} + \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V_{\text{ext}}|\psi|^2 + \frac{gN}{2}|\psi|^4 + \frac{N}{2}|\psi|^2\int V_g(|\vec{r}-\vec{u}|)|\psi(\vec{u},t)|^2\mathrm{d}^3u. \tag{5}$$

Now we come to the gravitational potential energy V_g appearing in eq. (5). In the framework of LED gravity, the hierarchy problem of the Standard Model can be ameliorated if (i) there exists a fundamental energy scale $M_*c^2 \approx 1$–1000 TeV (orders of magnitude less than the Planck energy $\cong 10^{19}$ GeV) for all interactions, and (ii) there are additional sub-millimetre scale spatial dimensions, so that the perceived weakness of Newtonian gravity on large scales in $(3+1)$-dimensional space–time is due to the gravitational field lines spilling into the hidden spatial dimensions[2,3]. In this formalism, the gravitational potential energy $V_g(r)$ between two point masses m_1 and m_2 separated by a distance r is given by

$$V_g(r) \cong -\frac{m_1 m_2}{m_{\text{pl}}^2}\frac{\hbar c}{r}, \quad r \gg R_*$$

$$\approx -\frac{(R_*(n))^n m_1 m_2}{m_{\text{pl}}^2}\frac{\hbar c}{r^{n+1}}, \quad r \ll R_*, \tag{6}$$

where $m_{\text{pl}} \equiv (\hbar c/G)^{1/2}$ is the Planck mass corresponding to the standard Newton's gravitational constant G and $R_*(n)$ is the radius of the extra dimensional n-torus given by

$$R_*(n) = \left(\frac{m_{\text{pl}}}{M_*}\right)^{2/n}\frac{\hbar}{2\pi M_* c}, \tag{7}$$

for $n = 1, 2, \ldots$. According to eq. (6), the closer one probes stronger is the gravity on scales smaller than $R_*(n)$. In the next section, we examine its implications on low-lying excitations of BEC.

Variational method, gravity and BEC oscillation modes

Solving eq. (3) with V_g given by eq. (6) is a nontrivial task. Instead, we take recourse to a variational method developed to study stability and low-energy excitations of BEC[10–12]. In this approach, the parameters of a trial wavefunction ψ_{tr} are obtained by demanding that the action is extremized by ψ_{tr}. Since attractive contact interactions (i.e. $a < 0$) are known to cause collapse of BEC[11,13] for sufficiently large N, stability analysis with gravitational interactions included is worth studying.

For this purpose, we consider a spherically symmetric trap potential

$$V_{\text{ext}} = \frac{1}{2}mw_0^2 r^2, \tag{8}$$

and choose a normalized trial wavefunction[11]

$$\psi_{\text{tr}}(\vec{r},\, t) = A(t)\exp(-r^2/2\sigma^2(t))\exp(\mathrm{i}B(t)r^2), \tag{9}$$

where $A(t)$, $\sigma(t)$ and $B(t)$ are amplitude, width and phase parameters respectively, that need to be determined from extremization of the action (eqs (4) and (5)). As ψ_{tr} is normalized, $A(t)$ and $\sigma(t)$ are related by

$$|A(t)|^2 = (\sqrt{\pi}\sigma(t))^{-3}, \tag{10}$$

so that

$$A(t) = (\sqrt{\pi}\sigma\,(t))^{-3/2}\exp(\mathrm{i}\gamma(t)), \tag{11}$$

where $\gamma(t)$ is a time-dependent phase. Substitution of eqs (8)–(11) in eq. (5) and carrying out the spatial integral thereafter leads to the following Lagrangian

$$L = \int d^3 r \mathcal{L} = \hbar\dot{\gamma} + L_{\text{int}} + \frac{gN}{4\sqrt{2}\pi^{3/2}\sigma^3} + \frac{3}{2}\sigma^2\left[\hbar\dot{B} + \frac{2\hbar^2}{m}B^2 + \frac{\hbar^2}{2m\sigma^4} + \frac{1}{2}mw_0^2\right], \quad (12)$$

where the gravity term is

$$L_{\text{int}} \equiv \frac{N}{2}\int d^3 r|\psi(\vec{r},t)|^2 \int V_g(|\vec{r}-\vec{u}|)|\psi(\vec{u},t)|^2 d^3 u. \quad (13)$$

Using eq. (6) for V_g, the above integral can be evaluated analytically for $n = 0$ and $n = 1$ cases so that

$$L_{\text{int}} = -\frac{\alpha_0}{\sigma} \quad \text{for } n = 0, \quad (14)$$

$$= -\frac{\alpha_1}{\sigma^2} \quad \text{for } n = 1, \quad (15)$$

where

$$\alpha_0 \equiv \frac{N\hbar c}{\sqrt{2\pi}}\left(\frac{m}{m_{\text{pl}}}\right)^2, \quad (16)$$

$$\alpha_1 \equiv \frac{NR_*\hbar c}{2\sqrt{2}}\left(\frac{m}{m_{\text{pl}}}\right)^2 \sum_{k=0}^{\infty}\frac{1}{k!}\frac{(2k+1)!!}{2^{2k}(2k+1)} \cong \frac{NR_*\hbar c}{2}\left(\frac{m}{m_{\text{pl}}}\right)^2. \quad (17)$$

Extremizing the action entails Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) = 0,$$

for $j = 1$ and 2, with $q_1 \equiv B$, $q_2 \equiv \sigma$ and L given by eq. (12) ($\gamma(t)$ is non-dynamical as it appears only as an additive total derivative term in eq. (12)). The equations of motion are

$$B(t) = \frac{m}{2\hbar}\frac{\dot{\sigma}}{\sigma}, \quad (18)$$

and

$$\hbar\dot{B} + \frac{2\hbar^2}{m}B^2 - \frac{\hbar^2}{2m\sigma^4} - \frac{gN}{4\sqrt{2}\pi^{3/2}\sigma^5} + \frac{1}{2}mw_0^2 = -f_n(\sigma), \quad (19)$$

where

$$f_n(\sigma) = \frac{\alpha_0}{3\sigma^3} \quad \text{for } n = 0,$$

$$= \frac{2\alpha_1}{3\sigma^4} \quad \text{for } n = 1. \quad (20)$$

By combining eqs (18) and (19), one arrives at the relevant equation needed to study small-amplitude oscillations in an ultra-cold cloud of bosons

$$m\ddot{\sigma} = -mw_0^2\sigma + \frac{\hbar^2}{m\sigma^3} + \frac{gN}{2\sqrt{2}\pi^{3/2}\sigma^4} - 2\sigma f_n(\sigma). \quad (21)$$

Employing the following dimensionless quantities[11] that make use of the BEC ground state scale $\sqrt{\hbar/mw_0}$

$$v \equiv \frac{\sigma}{\sqrt{\hbar/mw_0}}, \ \tau \equiv w_0 t, \ P \equiv \sqrt{\frac{2}{\pi}}\frac{Na}{\sqrt{\hbar/mw_0}},$$

$$\Rightarrow \frac{gN}{2\sqrt{2}\pi^{3/2}m} = \frac{\hbar^2}{m^2}\sqrt{\hbar/mw_0}\,P, \quad (22)$$

along with eq. (20) in eq. (21), we obtain

$$\frac{d^2 v}{d\tau^2} = -v + \frac{1}{v^3} + \frac{P}{v^4} + F_n(v), \quad (23)$$

for $n = 0, 1$, where the dimensionless gravitational accelerations have the forms

$$F_0(v) = -\sqrt{\frac{2}{3\pi^2}}N\left(\frac{m}{m_{\text{pl}}}\right)^2\left(\frac{c}{w_0\sqrt{\hbar/mw_0}}\right)v^{-2}, \quad (24)$$

and

$$F_1(v) = -\frac{2}{3}N\left(\frac{m}{m_{\text{pl}}}\right)^2\left(\frac{R_*}{\hbar/mc}\right)v^{-3}. \quad (25)$$

The RHS of eq. (23) corresponds to an effective potential $\Phi_n(v)$ ($n = 0, 1$) given by

$$\Phi_n(v) = \frac{1}{2}\left[v^2 + \frac{1}{v^2}\right] + \frac{P}{3v^3} + vF_0(v) \quad \text{for } n = 0, \quad (26)$$

$$= \frac{1}{2}\left[v^2 + \frac{1}{v^2}\right] + \frac{P}{3v^3} + \frac{vF_1(v)}{2} \quad \text{for } n = 1. \quad (27)$$

In order to study small-amplitude oscillation modes, one needs to find the minima of $\Phi_n(v)$. So, from $\Phi_n'(v) = 0$, the task here boils down to determining the zeroes of the quintic polynomial

$$v^5 - v - P - v^4 F_n(v) = 0. \quad (28)$$

To estimate numerically the real positive roots v_0 of eq. (28) and the excitation frequencies proportional to $\sqrt{\Phi_n''(v_0)}$, we make use of typical experimental length scales

$$\sqrt{\hbar/mw_0} \approx 10^{-4} \text{ cm}; \ (c/w_0)(m/m_{\text{pl}})^2 \approx 3\times10^{-26} \text{ cm};$$

$$a \approx 10^{-6}\text{cm}; \ \ R_*(1) \approx 200\ \mu\text{m}; \ \hbar/(mc) \approx 1.6\times10^{-16} \text{ cm}, \quad (29)$$

having in mind a BEC comprising ^{133}Cs for which $(m/m_{\text{pl}})^2 \cong 4.9 \times 10^{-34}$.

Since both P and $-F_n(v)$ increase with N, with the latter being negligibly smaller by orders of magnitude due to the smallness of $(m/m_{\text{pl}})^2$ in spite of the other factors (see eqs (22), (24), (25) and (29)), it is obvious that the *s*-wave scatterings completely swamp the gravitational corrections to the excitation frequencies. The oscillation modes of such a problem in the absence of gravity have already been studied by Perez-Garcia *et al.*[11].

To circumvent the dominance of binary *s*-wave scattering one may, along with augmenting N, invoke Feshbach resonance[14–17]. This effect enables experimentalists to tune the scattering length a magnetically, and reduce it to zero. Hence, with a vanishing P, in the $n = 0$ case (i.e. pure Newtonian gravity), one finds that for $N < 10^{21}$, the real positive root v_0 of eq. (28) is very close to unity corresponding to a frequency of $\omega = 2w_0$, as though the presence of $F_0(v)$ did not matter.

However, for macroscopically large occupation numbers $N = 10^{22}$ and 10^{23} (BECs of the future), one finds significant departures: $v_0 = 0.78$, $\omega = 2.4w_0$ and $v_0 = 0.12$, $\omega = 66w_0$ respectively. Because of the $3/v_0^4$ term in $\Phi_n''(v_0)$, one expects a higher excitation frequency as v_0 becomes smaller than unity. Although these results suggest that rise in self-gravity due to increase in N beyond 10^{22} makes the ultra-cold gas cloud shrink drastically, caution needs to be exercised in concluding so. For, when the number density $\approx N(\sqrt{\hbar/mw_0}\,v_0)^{-3}$ becomes very large, other subatomic effects will start dominating and, also, it is likely that the variational method demands more care in such circumstances. For instance, when $N = 10^{22}$, our result $v_0 = 0.78$ implies a mean separation between atoms in the condensate to be about 10^{-11} cm. Nevertheless, the observed pathology for $N \geq 10^{22}$ situation suggests that it would be interesting to study the numerical solutions of GPE, with Newtonian gravity added, for macroscopic BEC.

In the $n = 1$ case ($1/r^3$ gravity), when $P = 0$, the non-zero roots of eq. (28) satisfy

$$v_0^4 = 1 - (2N/3)(m/m_{\text{pl}})^2 (R_* /(\hbar / mc)), \quad (30)$$

implying that the roots are complex when

$$N > N_{\text{cr}} \equiv (3/2)(m/m_{\text{pl}})^{-2}(R_*/(\hbar / mc))^{-1}. \quad (31)$$

This is easily understood given that the potential Φ_1 of eq. (27) can be expressed as

$$\Phi_1(v) = \frac{1}{2}v^2 + \frac{1}{2v^2}\left[1 - \frac{N}{N_{\text{cr}}}\right], \quad (32)$$

provided a has been magnetically tuned to zero. From eq. (32), it is clear that the potential is no longer bounded from below when the occupation number exceeds N_{cr}.

This signals instability for the BEC since its size characterized by $\sigma(t)$ rolls down towards 0 as it tries to lower its potential energy. From the values provided in eq. (29), the onset of instability starts at $N_{\text{cr}} = 2.4 \times 10^{19}$. While, if $R_*(1)$ is smaller ≈ 1 µm, the critical occupation number for ^{133}Cs rises to $\approx 5 \times 10^{21}$. However, when $N < N_{\text{cr}}$, there is one positive root of eq. (30), and the corresponding excitation frequency is $2w_0$, albeit independent of $n = 1$ gravity.

Conclusion

Within the ambit of the variational method, we have found that occupation numbers in excess of N_{cr} cause collapse of BEC for attractive gravity falling off as r^{-3}. This can be subjected to experimental verification only when one attains BECs with macroscopically large occupational numbers $\approx 10^{19}$–10^{22}. For higher values of N, even Newtonian gravity appears to have significant effect on the BEC dynamics that needs to be studied more carefully. The consequences of $n \geq 2$ LED theories on BEC, though not covered in this article, need to be studied. In particular, it would be interesting to see whether their effects could be disentangled from those arising from other atomic interactions like van der Waals force.

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