A class of distance-based incompatibility measures for quantum measurements

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We discuss a recently proposed class of incompatibility measures for quantum measurements, which is based on quantifying the effect of measurements of one observable on the statistics of the outcome of another. We summarize the properties of this class of measures, and present a tight upper bound for the incompatibility of *any* set of projective measurements in finite dimensions. We also discuss non-projective measurements, and give a non-trivial upper bound on the mutual incompatibility of a pair of Lüders instruments. Using the example of incompatible observables that commute on a subspace, we elucidate how this class of measures goes beyond uncertainty relations in quantifying the mutual incompatibility of quantum measurements.

Keywords: Entropic uncertainty relation, fidelity, incompatibility, maximal disturbance.

Introduction

THE existence of incompatible observables in quantum theory is crucial to realizing several quantum information theoretic tasks, including most quantum cryptographic protocols. Quantifying the mutual incompatibility of a set of quantum measurements is therefore a question of some interest, both in quantum foundations and in quantum information theory.

One approach for quantifying the incompatibility of a set of quantum observables is based on uncertainty relations. In particular, lower bounds on the average uncertainties associated with a set of observables, obtained in the form of variance-based[1] or entropic[2] uncertainty relations, are often thought to provide an appropriate measure of incompatibility. However, this approach does not yield an incompatibility measure valid for *all* sets of observables, since the lower bound on the average uncertainty vanishes even when the observables in question are not compatible, but share a single common eigenstate.

This has motivated the study of operational measures of incompatibility that go beyond uncertainty relations. One such measure based on the idea of accessible fidelity[3], for example, quantifies the incompatibility of a set of observables as manifest in the *nonorthogonality* of their eigenstates[4].

In this article we discuss a different approach for quantifying incompatibility, based on estimating the *change* due to a measurement of one observable on the statistics of the outcomes of another[5]. If a pair of observables A and B does not commute, they are not jointly measurable. This implies that there exist states for which a measurement of A *disturbs* the system in such a way that a subsequent measurement of B yields probabilities that are different from those associated with a measurement of B alone. The *distance* between these two probability distributions – one resulting from a B-measurement following an A-measurement and the other resulting from a measurement of B alone – is indeed a measure of how the measurement of A affects the statistics of the outcomes of a measurement of B, for each given state. It was proposed that maximizing this over all the states of the system yields a measure of incompatibility that is naturally state-independent[5].

By choosing different measures of distance between probability distributions, a class of incompatibility measures is obtained. These measures indeed go beyond uncertainty relations in quantifying incompatibility – they always yield a strictly positive value even if the non-commuting observables in question share a common eigenstate, unlike uncertainty relations which give a zero bound in such cases. In other words, the distance-based incompatibility measures vanish iff the observables in question commute and are strictly non-zero otherwise.

The article is organized as follows. First, we briefly review the earlier approach of using uncertainty relations to quantify incompatibility in quantum theory. The distance-based incompatibility measures are then defined and their basic properties summarized. Tight upper bounds on the incompatibility measures are presented next. Exact expressions for the mutual incompatibility of a pair of qubit observables and for a specific example of incompatible observables that commute on a subspace are also discussed.

Quantifying incompatibility via uncertainty relations

The first quantitative statement on incompatibility of non-commuting observables was formulated in terms of

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variances for canonically conjugate variables[1]. In particular, for a pair of observables A and B, the Robertson–Schrödinger relation gives,

$$(\Delta_{|\psi\rangle}A)(\Delta_{|\psi\rangle}B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle|,$$

where $\Delta_{|\psi\rangle}X = \sqrt{\langle\psi|X^2|\psi\rangle - \langle\psi|X|\psi\rangle^2}$ $(X = A, B)$ is the variance associated with a measurement of X on *distinct* yet *identically prepared* copies of the state $|\psi\rangle$. Subsequently, it was proposed to quantify uncertainty using *entropic* quantities[2].

For an observable $A = \sum_i a_i P_i^A$ measured on state ρ, the probability distribution $\mathrm{Pr}_\rho^A \sim \{p_\rho^A(i)\}$ over the outcomes of the measurement is

$$\mathrm{Pr}_\rho^A : p_\rho^A(i) = \mathrm{tr}[P_i^A \rho].$$

For a set of measurements $\{A_1, A_2, ..., A_L\}$ with a finite set of outcomes, an entropic uncertainty relation (EUR) is a lower bound of the form

$$\left[\frac{1}{L}\sum_{j=1}^{L} S(A_j;\rho)\right] \geq c_S(\{A_j\}),\ \forall\rho,$$

where $S(A_j;\rho) = S(\{p_\rho^{A_j}(i)\})$ is a valid entropic function of the probability distribution $\mathrm{Pr}_\rho^{A_j}$. The lower bound $c_S(\{A_j\})$ is often thought of as a measure of the mutual incompatibility of the set of measurements $\{A_1, A_2, ..., A_L\}$.

There always exists a state ρ such that $S(A_j; \rho) = 0$ for one of the measurements A_j, namely, an *eigenstate* of A_j. Therefore, for a set of L observables in a d-dimensional space, the uncertainty lower bound satisfies

$$\left(1 - \frac{1}{L}\right)\log d \geq c_S(\{A_j\}) \geq 0.$$

If $c_S(\{A_j\}) = (1 - \frac{1}{L})\log d$, the set $\{A_j\}$ is *maximally incompatible*, implying a maximally strong uncertainty relation. However, $c_S(\{A_j\})$ is not a satisfactory measure of incompatibility for *all* sets of incompatible observables: it can attain a trivial (zero) value even when observables do not commute, whenever they have a single common eigenvector.

Distance-based incompatibility measures

An alternative, operational approach to quantifying incompatibility is based on estimating the *change* due to a measurement of one observable on the statistics of the outcomes of another which is measured subsequently. For a pair of observables A, B, we may consider the following two probability distributions. Let $\mathrm{Pr}_\rho^B \sim \{p_\rho^B(j)\}$ denote the probability distribution over the outcomes of a measurement of observable B in state ρ, and $\mathrm{Pr}_\rho^{A\to B} \sim \{q_\rho^{A\to B}(j)\}$ denote the probability distribution over the outcomes of a B measurement when it *follows* a measurement of A on the same state ρ. If A and B commute, then the two distributions are the same on *all* states.

However, if A, B do not commute, there exist states for which a measurement of A *disturbs* the system, so that $\mathrm{Pr}_\rho^{A\to B}$ and Pr_ρ^B are different. Maximizing the *distance* between $\mathrm{Pr}_\rho^{A\to B}$ and Pr_ρ^B over all the states of the system gives a measure of incompatibility that is naturally state-independent.

We consider the following well-known measures of distance[6] between a pair of discrete probability distributions $P \sim \{p_i\}$ and $Q \sim \{q_j\}$:

(i) Variational or L_1-distance:

$$D_1(P,Q) \equiv \frac{1}{2}\sum_i |p_i - q_i|.$$

(ii) Fidelity-based distance:

$$D_F(P, Q) \equiv 1 - (F(P, Q))^2,$$

where $F(P, Q)$ (fidelity or Bhattacharyya distance) is defined as $F(P, Q) \equiv \sum_i \sqrt{p_i}\sqrt{q_i}$.

(iii) Chebyshev or L_∞-distance:

$$D_\infty(P,Q) \equiv \max_i |p_i - q_i|.$$

All three measures satisfy

$$0 \leq D_\alpha(p,Q) \leq 1, \quad (\alpha \in \{1, F, \infty\}),$$

with $D_\alpha(P, Q) = 0$ if and only if P and Q are identical.

Corresponding to the distance measures, we are naturally led to the following *measures of incompatibility* of observable A with B[5]

(i) L_1-distance based incompatibility measure:

$$\mathcal{Q}_1(A \to B) = \sup_\rho D_1(\mathrm{Pr}_\rho^{A\to B}, \mathrm{Pr}_\rho^B).$$

(ii) Fidelity-based incompatibility measure:

$$\mathcal{Q}_F(A \to B) = \sup_\rho [1 - F^2(\mathrm{Pr}_\rho^{A\to B}, \mathrm{Pr}_\rho^B)].$$

(iii) L_∞-distance based incompatibility measure:

$$\mathcal{Q}_\infty(A \to B) = \sup_\rho D_\infty(\mathrm{Pr}_\rho^{A\to B}, \mathrm{Pr}_\rho^B).$$

All three incompatibility measures satisfy

$$0 \le \mathcal{Q}_\alpha(A \to B) \le 1, \ \alpha \in \{1, F, \infty\},$$

where the lower bound $\mathcal{Q}_\alpha(A \to B) = 0$ is attained iff A and B commute. The measures defined are not symmetric in general: there exist observables A, B, for which, $\mathcal{Q}_\alpha(A \to B) \ne \mathcal{Q}_\alpha(B \to A)$. The incompatibility $\mathcal{Q}_\alpha(A, B)$ of the pair of observables A, B, is therefore defined as

$$\mathcal{Q}_\alpha(A, B) \equiv \frac{\mathcal{Q}_\alpha(A \to B) + \mathcal{Q}_\alpha(B \to A)}{4}.$$

This ensures that $\mathcal{Q}_\alpha(A, B)$ is large when *both* $\mathcal{Q}_\alpha(A \to B)$ and $\mathcal{Q}_\alpha(B \to A)$ are large, and vice-versa. The incompatibility of a set of N observables $\{A_1, A_2, ..., A_N\}$ is therefore given by

$$\mathcal{Q}_\alpha(A_1, A_2, ..., A_N) \equiv \frac{1}{N^2} \sum_{i,j} \mathcal{Q}_\alpha(A_i \to A_j).$$

A relation between incompatibility and disturbance

For any observable $A \sim \{P_i^A\}$, the post-measurement transformation of state ρ after a measurement of A is described by a *CPTP* map $\mathcal{E}^A$, given by

$$\mathcal{E}^A(\rho) = \sum_i P_i^A \rho P_i^A.$$

The distance between the states $\mathcal{E}^A(\rho)$ and ρ is a valid measure of the *disturbance* caused to state ρ by a measurement of A[7,8].

The *maximal disturbance* due to the measurement of A can therefore be estimated by either of the following measures

$$\mathfrak{D}_1^{\max}(A) \equiv \sup_\rho \frac{1}{2} \operatorname{tr} |\mathcal{E}^A(\rho) - \rho|,$$
$$\mathfrak{D}_F^{\max}(A) \equiv 1 - [F^{(\min)}(A)]^2 = 1 - [\inf_\rho F(\mathcal{E}^A(\rho), \rho)]^2.$$

It has been shown that the incompatibility of A with B, as quantified by the measures $\{\mathcal{Q}_\alpha(A \to B)\}$, is always upper bounded by the maximal disturbance due to observable A[5].

Lemma 1. *For a pair of observables A and B with purely discrete spectra, the mutual incompatibility* $\mathcal{Q}_\alpha(A \to B)$ ($\alpha \in \{1, F, \infty\}$) *is bounded above by the maximal disturbance due to the measurement of A. That is*

$$\mathcal{Q}_\alpha(A \to B) \le D_1^{(\max)}(A), \ \alpha = 1, \infty,$$

$$\mathcal{Q}_F(A \to B) \le D_F^{(\max)}(A) = 1 - [F^{(\min)}(A)]^2.$$

The above relations between incompatibility and disturbance are a direct consequence of the following relations between the quantum distance measures and their classical counterparts[7]

$$D_1(\rho, \sigma) = \max_{\mathcal{M} \sim \{M_i\}} D_1(\Pr_\rho^{\mathcal{M}}, \Pr_\sigma^{\mathcal{M}}),$$

$$F(\rho, \sigma) = \min_{\mathcal{M}} F(\Pr_\rho^{\mathcal{M}}, \Pr_\sigma^{\mathcal{M}}),$$

where the optimization is over positive operator valued measures (POVMs).

Evaluating the incompatibility measures $\{\mathcal{Q}_\alpha\}$

Using the relation stated in Lemma 1, we may obtain upper bounds on the mutual incompatibility of any pair of observables. We state below the upper bound obtained for the fidelity-based incompatibility measure $\mathcal{Q}_F(A, B)$[5].

Theorem 2. *For a pair of observables A and B in a d-dimensional space, the mutual incompatibility of A and B is bounded by*

$$\mathcal{Q}_F(A, B) \le \frac{1}{2}\left(1 - \frac{1}{d}\right).$$

The upper bound is attained iff A and B are non-degenerate observables associated with mutually unbiased bases.

Recall that a pair of non-degenerate observables $A \sim \{|a_i\rangle\}$ and $B \sim \{|b_j\rangle\}$ is said to be mutually unbiased iff $|\langle a_i|b_j\rangle|^2 = 1/d, \forall i, j$.

Theorem 2 has the following important corollary: the average pairwise mutual incompatibility of a set of N observables $\{A_1, A_2, ..., A_N\}$ in a d-dimensional space is bounded by

$$\mathcal{Q}_F(A_1, A_2, ..., A_N) \le \left(1 - \frac{1}{N}\right)\left(1 - \frac{1}{d}\right).$$

The bound is attained iff the observables are non-degenerate and associated with mutually unbiased bases.

It is easy to see that the measures $\mathcal{Q}_1(A \to B)$ and $\mathcal{Q}_\infty(A \to B)$ also attain the same value for a pair of mutually unbiased observables

$$\mathcal{Q}_1(A \to B) = \mathcal{Q}_\infty(A \to B) = 1 - \frac{1}{d}.$$

It would therefore seem reasonable to conjecture that both $\mathcal{Q}_1(A, B)$ and $\mathcal{Q}_\infty(A, B)$ are also bounded above by $\frac{1}{2}(1-\frac{1}{d})$, for *any* pair of observables in a d-dimensional space.

Incompatibility of qubit observables

Evaluating the incompatibility of a general set of observables involves solving a hard optimization problem. However, all three measures $\mathcal{Q}_1$, $\mathcal{Q}_\infty$ and $\mathcal{Q}_F$ can be evaluated exactly for a pair of qubit observables[9]. Consider a pair of observables A, B on a two-dimensional space with corresponding Bloch sphere representations $A = \alpha_1\mathbb{I} + \alpha_2\,\vec{a}\cdot\vec{\sigma}$ and $B = \beta_1\mathbb{I} + \beta_2\,\vec{b}\cdot\vec{\sigma}$, where $\vec{a},\vec{b} \in \mathbb{R}^3$ are unit vectors and $\{\alpha_i, \beta_i\} \in \mathbb{R}$. Using this parameterization, it is possible to show that

$$\mathcal{Q}_\infty(A \to B) = \mathcal{Q}_1(A \to B) = \frac{1}{2}\sqrt{1-(\vec{a}\cdot\vec{b})^2},$$

$$\mathcal{Q}_F(A \to B) = \frac{1}{2}(1-(\vec{a}\cdot\vec{b})^2). \quad (1)$$

As expected, all three measures coincide for the limiting cases. That is, (a) when A and B commute, $(\vec{a}\cdot\vec{b})^2 = 1$ and all three measures give 0, and, (b) when A and B are mutually unbiased, $\vec{a}\cdot\vec{b} = 0$, and $\mathcal{Q}_1(A \to B) = \mathcal{Q}_\infty(A \to B) = \mathcal{Q}_F(A \to B) = 1/2$. For any other pair of qubit observables, the fidelity-based measure $\mathcal{Q}_F(A, B)$ is in general smaller than $\mathcal{Q}_1(A, B)$ and $\mathcal{Q}_\infty(A, B)$.

Non-projective measurements: incompatibility of a pair of Lüders instruments

The measures of incompatibility defined above can also be extended to the case of general quantum measurements, beyond the class of projective measurements. Consider the class of POVMs $\mathcal{A}$ with discrete outcomes described by a collection of positive operators $\{0 \le A_i \le \mathbb{I}\}$ satisfying $\sum_i A_i = \mathbb{I}$. One simple implementation of a measurement of a POVM $\mathcal{A}$ is given by the so-called *Lüders instrument* $\Phi_{\mathcal{L}}^{\mathcal{A}}$, in which the post-measurement state after a measurement of observable $\mathcal{A}$ on state ρ is given by[10]

$$\Phi_{\mathcal{L}}^{\mathcal{A}}(\rho) = \sum_{i=1} A_i^{1/2}\rho A_i^{1/2}.$$

The incompatibility of a pair of POVMs $\mathcal{A}$ and $\mathcal{B}$, with finite number of outcomes N_A and N_B, and corresponding Lüders channels

$$\Phi_{\mathcal{L}}^{\mathcal{A}}(\rho) = \sum_{i=1}^{N_A} A_i^{1/2}\rho A_i^{1/2};\ \Phi_{\mathcal{L}}^{\mathcal{B}}(\rho) = \sum_{j=1}^{N_B} B_j^{1/2}\rho B_j^{1/2},$$

can be shown to be bounded by[5]

$$\mathcal{Q}_F(\Phi_{\mathcal{L}}^{\mathcal{A}} \to \Phi_{\mathcal{L}}^{\mathcal{B}}) \le 1-\frac{1}{N_A}.$$

Observables that commute on a subspace

Finally, we consider an example which shows clearly that the class of measures $\{\mathcal{Q}_\alpha\}$ goes beyond uncertainty relations in quantifying incompatibility. Consider a pair of non-degenerate observables A, B that commutes over a subspace of dimension d_c, such that, A, B share d_c common eigenvectors, and are mutually unbiased in the $(d - d_c)$ dimensional subspace where they do not commute.

In other words, the eigenstates $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ of A and B satisfy

$$|a_i\rangle = |b_i\rangle, \quad \forall i = 1,\ldots,d_c,$$

$$|\langle a_i | b_j\rangle| = \begin{cases} 0 & \text{for } i \le d_c,\, j > d_c \\ 0 & \text{for } i > d_c,\, j \le d_c \\ \dfrac{1}{\sqrt{d-d_c}} & \text{for } i, j > d_c \end{cases}.$$

The mutual incompatibility of A and B is then given by,

$$\mathcal{Q}_F(A, B) = \frac{1}{2}\left(1-\frac{1}{d-d_c}\right).$$

Clearly, $\mathcal{Q}_F(A, B) > 0$ for $0 \le d_c < d - 1$. On the other hand, the entropic uncertainty lower bound vanishes for such a pair of observables, for any $d_c > 0$. Interestingly, even optimal entropic uncertainty relations formulated for the successive measurement scenario yield a trivial lower bound of zero, when the observables in question share a single eigenvector[11].

Summary

We have summarized a novel approach to quantify the mutual incompatibility of quantum observables, in terms of the *change* caused by a measurement of one observable on the *statistics* of the outcomes of a subsequent measurement of the other observable. The class of measures discussed here is indeed distinct from the incompatibility measure defined in Bandyopadhyay and Mandayam[4] based on the accessible fidelity, though all measures coincide for the limiting cases of commuting and mutually unbiased observables. The operational setting motivating these measures is a commonly encountered one in the context of quantum cryptography, and it is the subject of ongoing work to see if these measures can play a direct role in analysing the security of quantum cryptographic protocols. While the incompatibility measures $\{\mathcal{Q}_\alpha\}$ are hard to evaluate in general, recent investigations show that non-trivial lower bounds can be obtained[9], which are efficiently computable using convex optimization techniques[12].

1. Heisenberg, W., Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik. *Z. Phys.*, 1927, **43**(3–4), 172–198; Robertson, H. P., The uncertainty principle. *Phys. Rev.*, 1929, **34**(1), 163–164.
2. Wehner, S. and Winter, A., Entropic uncertainty relations – a survey. *New J. Phys*., 2010, **12**(2), 025009-1 to 22.
3. Fuchs, C. A. and Sasaki, M., Squeezing quantum information through a classical channel: measuring the quantumness of a set of quantum states. *Quant. Inf. Comput.*, 2003, **3**(5), 377–404.
4. Bandyopadhyay, S. and Mandayam, P., Operational measure of incompatibility of noncommuting observables. *Phys. Rev. A*, 2013, **87**, 042120-1 to 6.
5. Mandayam, P. and Srinivas, M. D., Measures of disturbance and incompatibility for quantum measurements. *Phys. Rev. A*, 2014, **89**(6), 062112-1 to 9.
6. Mathai, A. M. and Rathie, P. N., *Basic Concepts in Information Theory and Statistics*, Wiley Eastern, New Delhi, 1975.
7. Nielsen, M. A. and Chuang, I. L., *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.
8. Mandayam, P. and Srinivas, M. D., Disturbance trade-off principle for quantum measurements. *Phys. Rev. A*, 2014, **90**(6), 062128-1 to 8.
9. Kundu, S., Wagh, K. and Mandayam, P., Quantifying incompatibility beyond entropic uncertainty; arxiv:quant-ph/1510.04093.
10. Heinosaari, T. and Ziman, M., *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement*, Cambridge University Press, Cambridge, 2012.
11. Srinivas, M. D., Optimal entropic uncertainty relation for successive measurements in quantum information theory. *Pramana*, 2003, **60**(6), 1137–1152.
12. Watrous, J., Semidefinite programs for completely bounded norms. *Theor. Comput*., 2009, **5**, 217–238.

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