Weak measurements: typical weak and superweak values

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Weak value is a physical property of a quantum system which manifests itself through a weak measurement using different pre- and post-selected ensembles of the system. The weak values of an operator may differ significantly from its eigenvalues and can even lie outside the spectrum if it is bound: they can be 'superweak'. The latter, originating due to a coherent superposition of waves, may appear as a 'supershift' on the measuring device. This property has potential application in the amplification and detection of extremely weak signals.

Keywords: Eigenvalue, ensemble, quantum system, weak measurement.

Introduction

THE dynamical laws are time-symmetric, both in classical as well as in quantum regime. For example, if the position and velocity of a classical system are known at a given instant t, its past (before t) and future (after t) can in principle be predicted from the laws of classical mechanics. Similarly, an isolated quantum system undergoes a deterministic, unitary, continuous time-evolution that is governed by the Schrödinger equation (or Dirac equation in relativistic limit) and its state and associated properties both in the past as well as the future can be obtained, in principle, by solving the equation. For any physical property (observable), the state of the system can be expressed as a linear combination of the eigenstates of the related operator; the observable then has a finite probability to take any one of the eigenvalues. In reality, however, the determination of a physical observable requires a direct or indirect measurement which leads to collapse of the state of the system to one of the eigenstates, with the observable given by the corresponding eigenvalue. After the collapse, the system again evolves according to the Schrödinger equation. But due to collapse, the state of the system in the past can no longer be determined, thus resulting in time-asymmetry. In other words, a measurement disturbs/alters the quantum system.

All measurements begin with an interaction between the system under consideration and a measuring device. The latter is simply another physical system, governed by the same set of physical laws as the one being measured. For quantum systems, however, the measurement is a two-way process: any such interaction must also disturb the system being measured. Consequently, even if the state of a system is known at time t, and its Hamiltonian is known at all times, the result of a measurement performed at a later time t_1 or previous time t_0 cannot be exactly predicted. Thus both 'past' and 'future' of a quantum system are equivalent (equally good or bad) as far as prediction of the 'present' is concerned. This then leads to the question: Why should one always express the present state in terms of the past one only? Why not use the evolution from both states, past (or pre) as well as future (or post), which at least preserves time-symmetry? This idea of 'two state' quantum mechanics[1] led to the introduction of a new concept, namely, 'weak values', which are the values obtained for an observable on its 'weak measurement'. The latter, a technique invented by Aharonov and co-workers about 20 years ago, is a way of probing a quantum system which minimizes the measurement disturbance[2–6]. The most interesting feature of a weak measurement is revealed when it is subjected to conditions involving pre-selection and post-selection of the state of the quantum system: for a post-selection with very low probability (that is, if the measurement is conditioned on finding the system in a very unlikely final state), the measurement outcome can be unexpectedly large, larger than any expected value for the property being measured.

For a clear understanding of how weak values arise in a weak measurement, it is imperative to first review the standard measurement techniques. An ideal measurement of an observable A requires that the state of the system remains unchanged after the measurement: if before measurement, the system is in an eigenstate of A with an eigenvalue a_n, the outcome of the measurement is a_n. To understand how the eigenvalue appears on the measuring device, one has to consider its interaction with the system.

In standard measurement procedure of a variable A of a quantum system, if Q refers to the pointer variable of the measuring device, then its conjugate variable P couples with A. The Hamiltonian for the interaction between the measuring device and the system can then be described by (von Neumann Hamiltonian)[2]

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$$H_{\text{int}}(t) = \lambda(t)AP, \tag{1}$$

where $\lambda(t)$ is the coupling function specifying the time of the measurement interaction. Clearly, for the state of the system to remain unchanged after the measurement, it is necessary that H_{int} is applicable for a very short time, that is, $\lambda(t)$ is non-zero only for a short time δt. As the Hamiltonian of the system (non-interacting part) is negligible during the time of measurement, the equation of motion for the pointer variable Q can be given as

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{i}{\hbar}[H_{\text{int}}, Q] = \lambda A. \tag{2}$$

It is easy to see from the above that the shift in position Q of the pointer gives the measured value of A

$$Q_{\text{f}} - Q_{\text{i}} = \lambda A, \tag{3}$$

with Q_{i} and Q_{f} as the initial and final positions of the pointer variable.

An ideal measurement therefore requires both Q_{i}, Q_{f} to be well-defined. In real measurements, however, the state of the pointer of the measuring device is given by a distribution, say $\Phi(Q)$ of width Δ. As a consequence, the final pointer state does not give a precise value of the observable A. This can be explained as follows. Consider the initial states of the system and measuring device before measurement to be $|i\rangle$ and $|\Phi\rangle$ respectively. The coupled state of the system + measuring device can then be given as

$$\mathrm{e}^{-(i/\hbar)\int H\mathrm{d}t}\,|\,i\rangle\,|\,\Phi\rangle = \mathrm{e}^{-(i/\hbar)\lambda PA}\,|\,i\rangle\,|\,\Phi\rangle. \tag{4}$$

The final pointer state $|\psi\rangle$ after the measurement corresponding to system-state $|i\rangle$ now becomes $|\psi\rangle = \langle i|\mathrm{e}^{-(i/\hbar)\lambda PA}\,|\,i\rangle\,|\,\Phi\rangle$, which in position representation Q can be written as

$$\psi(Q) = \langle i|\mathrm{e}^{-(i/\hbar)\lambda PA}\,|\,i\rangle\,\Phi(Q). \tag{5}$$

For a measurement of A, it is physically motivating to express the initial state $|i\rangle$ of the system in the eigenbasis of A. Consider A as an operator with bound spectrum, with a_n as the eigenvalues and $|a_n\rangle$ corresponding eigenfunctions ($n = 1, ..., N$). One can then write $|i\rangle = \sum_n c_n|a_n\rangle$ with c_n as the coefficients. This gives the final pointer state as a combination of many copies of the initial pointer state, each one centred around an eigenvalue

$$\begin{aligned}\psi(Q) &= \sum_n |\,c_n\,|^2 \mathrm{e}^{-\lambda a_n \partial_Q}\Phi(Q)\\ &= \sum_n |\,c_n\,|^2 \Phi(Q - \lambda a_n).\end{aligned} \tag{6}$$

As clear from the above, the final pointer state is a distribution, with error Δ, around mean $\lambda\sum_k|c_n|^2a_n$. Note the latter is just the expectation value of A in the state $|i\rangle$; thus the mean pointer position $\langle Q\rangle$ gives $\langle A\rangle$. Although the error in the final pointer state can be reduced by choosing $\Delta \to 0$ (or $\Delta \ll \Delta(a_n)$), it enhances the error associated with the momentum P (i.e. the conjugate variable of Q) of the pointer. This in turn increases the interaction between the system and the measuring device (see eq. (1)) and the system, after the measurement, is no longer the same as before. Consequently, the measurement necessarily disturbs the subsequent measurements of all observables, say B non-commuting with A: the result of measurement of B after A will be different from B before A (ref. 2).

As the objective of any measurement would be to leave the system unperturbed for subsequent measurements, it is necessary to keep the interaction between the measuring device and the system weak. Such a 'weak measurement', however, causes a large uncertainty in the pointer variable Q ($H_{\text{int}} \to 0 \to P \to 0 \to \Delta \to$ large $\gg$ spectrum length) and the state of pointer is then centred around $\langle A\rangle$ with spread Δ. Although one such measurement gives no information (as $\Delta \gg \langle A\rangle$), repeating it, say M times, on a given system reduces the uncertainty by $1/\sqrt{M}$. Alternatively, one can make the same measurement on each member of a sufficiently large ensemble of systems, all in a same initial state (and each connected to a separate measuring device). This will again reduce the uncertainty by $1/\sqrt{M}$ while keeping the average at $\langle A\rangle$. Enlarging the ensemble will lead to the measurement of $\langle A\rangle$ with any desired precision. Furthermore, as the measurements hardly disturb the ensemble, they characterize the ensemble during all intermediate times. And even non-commuting operators can be measured at the same time (as a single measurement is imprecise).

As mentioned above, the outcome of the measurement is sensitive to the size of the ensemble (consisting of exact replicas of the system). This leads to curiosity about the exact role of the ensemble: can the outcome be changed by averaging over Q-values of only a part of the original ensemble? In other words, what happens if the 'post-selected' ensemble which is considered to obtain the average over Q, is just a sub-ensemble of the original ensemble over which the measurement was performed? The original ensemble can therefore be referred as the 'pre-selected' ensemble. It is important to emphasize here that the 'post-selected' ensemble refers to a sub-ensemble of systems and not of measuring devices; (a sub-ensemble of the latter can always be chosen in a trivial way to change the outcome, e.g. considering only those measuring devices with large value of Q will maximize the outcome). In fact, as shown by Aharonov and collaborators[2], a suitable choice of pre- and post-selected ensembles can lead to values different from $\langle A\rangle$. This clearly indicates the lack of ergodicity in the ensemble of

systems which could be due to restrictions imposed by a uncertainty.

There is another reason to choose different pre- and post-selected ensembles: measurement via an initial state ensemble only breaks time-symmetry. The latter can be preserved by selecting both past as well as future of a state at time t, i.e. by choosing two ensembles of the system, one prior to and the other after the measurement. Measuring an operator A weakly at time t between a pre-selected state, say $|i\rangle$ at time $t_0 < t$ and a post-selected state, say $|f\rangle$ at $t_1 > t$ yields values that need not be eigenvalues or even classically allowed; these values, known as weak values, can be described by

$$A_{\mathrm{w}} = \frac{\langle f \mid A \mid i\rangle}{\langle f \mid i\rangle}. \tag{7}$$

It is worth noting that A_{w} is a complex number and is different from $\langle A\rangle$ if $|i\rangle$ and $|f\rangle$ are different. As mentioned above, the weak measurement of A at time t with an initial ensemble of size M at time t_0 gives the final pointer state centred around expectation value $\langle A\rangle$ with error $\Delta/\sqrt{M}$. The weak measurement, for both past and future ensemble chosen, replaces $\langle A\rangle$ by $\mathrm{Re}(A_{\mathrm{w}})$; here $\mathrm{Im}(A_{\mathrm{w}})$ gives the shift of the momentum of the pointer of the measuring device. In case of an operator with a bound spectrum, the weak-value measurement may lead to surprising, unexpected results: as well known, the expectation value $\langle A\rangle$ never lies outside the range of a bound spectrum, but the weak value (i.e. $\mathrm{Re}(A_{\mathrm{w}})$ or $\mathrm{Im}(A_{\mathrm{w}})$) can lie outside the spectrum range. *It can be a superweak value*. An example of the weak value is the local expectation value of an operator A at position r, which can be described as the real part of the weak value of A in state $|\psi\rangle$ with position $\mathbf{r}$ post-selected

$$A(r) = \frac{\langle \psi \mid \frac{1}{2}(\delta(\hat{\mathbf{r}}-\mathbf{r})\hat{A} + \hat{A}\delta(\hat{\mathbf{r}}-\mathbf{r})) \mid \psi\rangle}{\langle \psi \mid \delta(\hat{\mathbf{r}}-\mathbf{r}) \mid \psi\rangle}$$

$$= \mathrm{Re}\frac{\langle \mathbf{r} \mid \hat{A} \mid \psi\rangle}{\langle \mathbf{r} \mid \psi\rangle}. \tag{8}$$

The appearance of weak values in pointer-state distribution can be explained follows: after an impulsive measurement of A and projection onto a final system-state $|f\rangle$, the final state of the measuring device is

$$\psi(Q) = \langle f \mid \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_{\mathrm{int}}} \mid i\rangle \Phi(Q). \tag{9}$$

As H_{int}, given by eq. (1), is small for a weak measurement ($|P|$ is small for $\Delta \to$ large), one can approximate

$$\psi(Q) \approx \left\langle f \mid 1 - \frac{i\lambda}{\hbar} PA \mid i \right\rangle \Phi(Q)$$

$$\approx \langle f \mid i\rangle \exp\left[-\frac{i\lambda}{\hbar} PA_{\mathrm{w}}\right] \Phi(Q)$$

$$\approx \langle f \mid i\rangle \Phi(Q - \lambda A_{\mathrm{w}}), \tag{10}$$

with A_{w} given by eq. (7). As clear from the above, the final pointer state in this case is just the initial pointer state with a shifted centre and a modified amplitude; the shift is proportional to the weak value A_{w}. An important point to note here is that while the amplitude of $\psi(Q)$ is proportional to $\langle f \mid i\rangle$, the latter appears in the denominator of A_{w}. Thus although a suitable choice of post- and pre-selected states, e.g. (those nearly orthogonal mutually) makes A_{w} very large, it also makes the final pointer state very weak.

The sensitivity of the results of weak measurements to the choice of pre- and post-selected states makes it relevant to probe their role in more detail. An important question in this connection is as to how weak values are typically distributed if the pre- and post-selected states are random states? It is also important to know the influence, if any, of the type of randomness and whether superweak values are common or rare? Seeking the answers in Berry and Shukla[7,8], the probability distribution $P(A_{\mathrm{wr}})$ of $A_{\mathrm{wr}} \equiv \mathrm{Re}A_{\mathrm{w}}$ was calculated for random pre- and post-states in the eigenbasis of A (i.e. $|i\rangle = \sum i_n |a_n\rangle$ and $|f\rangle = \sum_n f_n |a_n\rangle$ with i_n and f_n randomly distributed); note this corresponds to considering an ensemble of states $|i\rangle$ and $|f\rangle$ which should not be confused with pre- and post-selected ensembles of systems, each one of which is at the same state $|i\rangle$ and $|f\rangle$ respectively. For arbitrary randomness, we find an unanticipated universality in the distribution of weak values. If there are many eigenvalues lying within a finite range, this distribution takes a simple generalized Lorentzian form

$$P(A_{\mathrm{wr}}) = \frac{\langle a_n^2\rangle}{2(\langle a_n^2\rangle + A_{\mathrm{wr}}^2)^{3/2}}. \tag{11}$$

The distribution $P(A_{\mathrm{wr}})$ of the imaginary part of weak values $A_{\mathrm{wi}} \equiv \mathrm{Im}A_{\mathrm{w}}$ also turns out to be analogous. This gives the superweak probability of the weak value lying outside the spectrum of A as

$$P_{\mathrm{super}} = \int_{-\infty}^{a_{\min}} \mathrm{d}A_{\mathrm{wr}} P(A_{\mathrm{wr}}) + \int_{a_{\max}}^{\infty} \mathrm{d}A_{\mathrm{wr}} P(A_{\mathrm{wr}})$$

$$= 2\int_{a_{\max}}^{\infty} \mathrm{d}A_{\mathrm{wr}} P(A_{\mathrm{wr}})$$

$$= 1 - \frac{a_{\max}}{\sqrt{\langle a_n^2\rangle + a_{\max}^2}}. \tag{12}$$

The above indicates the universality of the superweak probability and as shown in Berry and Shukla[7] by the examples of interesting eigenvalue distributions, e.g. uniform, semicircle, bimodal, etc. the superweak probability of weak values lying outside the spectrum can be as large as 0.293 (almost 30% chance of a weak value being superweak). By contrast, the familiar expectation values always lie within the spectral range, and their distribution, although approximately Gaussian for many eigenvalues, is not universal[7].

As indicated by eq. (11), the distribution of weak values for large spectra has a generalized universal form, independent of size N of a spectrum, its nature (whether eigenvalues are randomly distributed or regularly arranged), and, of the nature of pre- and post-selected ensembles. Note $P(\mathrm{Re}A_w)$ and $P(\mathrm{Im}A_w)$ for large N are similar and smooth functions in which the only indication of the extent of the spectrum of the observable A is a scaling variable, a measure of the distribution of N eigenvalues. By contrast, for $N = 2$, $P(\mathrm{Re}A_w)$ and $P(\mathrm{Im}A_w)$ are different and can be discontinuous too. But the distributions for both small ($N = 2$) and large N decay in the same way for large $|A_w|$ (as $|A_w|^{-3}$). The superweak probabilities in the two cases are also not very different: for large N, P_{super} can be as large as 0.293..., for $N = 2$, $P_{\mathrm{super}} = 1/3$. An interesting point to note here is that P_{super} is the same as the super-oscillation probability for Gaussian random monochromatic waves in two dimensions[10].

A weak value as well as a superweak value result from a conspiracy of pre- and post-selected states and the operator A. Mathematically the origin of superweak value can be explained using the concept of super-oscillation of functions. The latter is defined as the oscillations in a band-limited function faster than the maximum frequency over arbitrarily large intervals. This leads to a natural query whether a superweak value always shows up on a measuring device and under what conditions will it lead to a super-shift of the pointer? The latter can be explained as follows: from eq. (10), final state Φ of the pointer is a superposition of the shifted copies of the initial pointer state. A coherent interference can give rise to a reproduction of Φ centred far away from any of the copies, albeit greatly reduced in strength: *this is called a supershift*. As discussed in Berry and Shukla[9], the occurrence of supershift is a bigger conspiracy; it not only depends on the choice of pre- and post-selected states but also on Δ, the uncertainty associated with the initial pointer state. It requires pointer state $\phi(q)$ to be broad enough, i.e. $\phi(p)$ must decay fast enough, not to kill super-oscillations in the Fourier transform of the overlap $\langle f|i\rangle$ of the pre–post pair. The analysis discussed in Berry and Shukla[9] indicates that although superweak value exists for the pre–post pair, the supershift can be seen only for the Gaussian and Lorentzian shapes of $\phi(q)$ but not for exponential shapes. In other words, the supershift is a kind of 'resurrection from the dead' involving the tail of $\phi(q)$; it may or may not happen, depending on the interplay of the analytic form of the tail of $\phi(q)$ and the chosen pre-, and post-states.

It is worth considering the relevance of weak and superweak values from an application point of view. A measurement result is preferable if it is independent of the choice of the ensemble used to reduce the error introduced by a single weak measurement. The independence of the distribution of weak values from the nature of the pre- and post-selected ensembles, in contrast to ensemble dependence of the distribution of expectation values, suggests weak values as a better tool for the measurement. Weak value technique can also be used for amplification of extremely weak signals. But a sacrifice is necessary to achieve amplification: it comes in the form of throwing away most of the data in the post-selection process. This has already been applied successfully in many experiments. For example, the technique has been used by Hosten and Kwiat[11] to amplify the displacement of a laser beam by a factor of thousand. This allowed them to measure the displacement of 0.1 nm and thus confirm the existence of spin Hall effect for light. The technique also helped in the detection of a spin-dependent displacement perpendicular to the refractive index gradient for photons passing through an air–glass interface; this in turn indicates universality of the Hall effect for particles of a different nature. Using the technique along with a Sagnac ring interferometer, Ben Dixon *et al.*[12] were able to detect very small transverse deflections (of order 10 fm) of an optical beam. Recently, the weak value concept was applied to explore the possibility of the apparent superluminal velocity of the neutrinos as a superweak value of the velocity difference operator[13]. Weak values can assist in understanding many counterintuitive quantum results: they can be used as a fundamental test of quantum mechanics by ruling out a class of macro-realistic hidden variable theories, and are equivalent to the violation of generalized Leggett–Garg inequalities[14], help resolve quantum paradoxes such as Hardy paradox[15], apparent superluminal travel[16], and counterfactual problems such as the three-box problem.

In the end, we emphasize that both the eigenvalues as well as weak values are fundamental aspects of quantum systems; they exist, one measures them or not. But weak value is a normalized matrix element of an operator between $|\mathrm{pre}\rangle$ and $\langle\mathrm{post}|$ whose normalization with $\langle\mathrm{post}|\mathrm{pre}\rangle$ only takes meaning because of the uncertainty in measuring device. Further a conventional 'strong' measurement done with pre/post state will never give a superweak value, although it may exist. A weak measurement can give a superweak value only if the measuring device uncertainty is large enough so as not to kill the super-oscillatory behaviour of the pre- and post-state combination.

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