The uncertainty relations in quantum mechanics

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The notion of uncertainty in the description of a physical system has assumed prodigious importance in the development of quantum theory. Overcoming the early misunderstanding and confusion, the concept grew continuously and still remains an active and fertile research field. Curious new insights and correlations are gained and developed in the process with the introduction of new 'measures' of uncertainty or indeterminacy and the development of quantum measurement theory. In this article we intend to reach a fairly uptodate status report of this yet unfurling concept and its interrelation with some distinctive quantum features like nonlocality, steering and entanglement/inseparability. Some recent controversies are discussed and the grey areas are mentioned.

Keywords: Complementarity, entanglement, harmonic analysis, uncertainty relations, wave–particle duality.

IN March 1927, following a heuristic approach and applying some plausible measure for what he called inaccuracy or indeterminacy in the measurement of a physical quantity, Heisenberg[1] arrived at his celebrated uncertainty relations (note 1). The famous thought experiment in which Heisenberg imagined measuring the position of an electron using a gamma-ray microscope, finally leads to the concept of a minimum uncertainty product (of the position and momentum uncertainties of the electron under observation) of the order of the Planck's constant. He also pointed out that the position–momentum uncertainty relation 'is a precise expression for the facts which one earlier sought to describe the division of phase space into cells of magnitude h' and summarized the import of the relation as 'Too much precision in q_0 (initial position coordinate) produces great uncertainty in p_0 (initial momentum)'.

Though Heisenberg asserted that, 'this relation is a straight forward mathematical consequence of the quantum mechanical commutation rule for the position and the corresponding momentum operators $\mathbf{pq} - \mathbf{qp} = -i\hbar$', he actually derived the relation using a semi-quantitative definition of imprecision/indeterminacy in the position coordinate q and the corresponding momentum p in terms of 'spreading' of the Gaussian 'probability–amplitude packet' of a microparticle (like electron) and dubbed it 'a slight generalization' of the Dirac–Jordan formulation. He then proceeded to discuss 'a few special idealized experiments' for simultaneous measurement of position and the conjugate momentum and also of time and energy using specially the Stern–Gerlach experiment and an example of the energy and phase of the state of an atom. However, it should be noted that Heisenberg's view on time was both ambiguous and contradictory[2]. In the passing we also note that, although the time–energy uncertainty is one of the most important relations apart from the position–momentum uncertainty and its heuristic meaning (note 2; ref. 3) was readily appreciated by the early founders of quantum mechanics (QM), both its formulation and interpretation are different from that of the position–momentum uncertainty and require special consideration. The problem is that the time is not usually recognized as a dynamical variable/an operator in standard quantum mechanics (note 3). We shall revert to discuss aspects of the time–energy uncertainty and its formulation in the following.

As the title of Heisenberg's paper (translated in English as: On the physical/intuitive content of the quantum theoretic kinematics and mechanics) suggests, the purpose of the formulation was to bring out explicitly some important features of QM not quite obvious from the formalism itself. Also, he did not give a general definition for the 'uncertainties' and quantifies them only on a case-to-case basis with the remark that they could be taken as 'something like the mean error'. Apparently, when Heisenberg refers to the uncertainty or indeterminacy of a quantity, he means that the value of this quantity cannot be given beforehand.

According to Heisenberg[1], the uncertainty relations created 'room' (1927, p. 180) or 'freedom' (1931, p. 43) for the introduction of some non-classical mode of description of experimental data. Although Bohr accepted the conclusions of the paper, he disagreed with Heisenberg's conception of indeterminacy as a limitation of the applicability of classical notions. In fact, Heisenberg's paper contained an 'Addition in proof' mentioning critical comments by Bohr, who pointed out that in the γ-ray microscope experiment it is not the change of momentum of the electron due to a position measurement that is important, but the circumstance that this change cannot be precisely determined in the same experiment.

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For Bohr, the central idea was wave–particle duality and in spite of the fact that the views on QM of the two founding fathers, Heisenberg and Bohr, are often clubbed as the 'Copenhagen interpretation', there is considerable difference between their views on uncertainty relation, wave–particle duality and Bohr's complementarity principle (BCP, to be discussed later). Bohr has criticized Heisenberg for his suggestion that these relations were due to discontinuous changes occurring during the process of measurement and pointed out that the uncertainties in the experiment did not exclusively arise from the discontinuities (existence of quantum of action), but also from the fact that the position and the momentum of the electron cannot be simultaneously defined in the microscope experiment ('Addition in proof' in Heisenberg[1]) and we need to consider both the particle theory and the wave theory. On the other hand, Bohr has always defended the uncertainty relations against the objections raised by Einstein in his famous thought experiments christened as Einstein's slit and Einstein's box[3]. In fact, they spent many years intensely debating the 'picture of reality' as described by the quantum mechanical description.

Although Heisenberg's thought experiments are considered in most textbooks on QM as the 'experimental' or 'logical' foundations, they were open to criticisms that they tacitly assume the classical ontology which they explicitly purport to deny (Jammer[4], p. 73). It is also important to note that though the description of standard QM (the negation of a phase-space description at the outset) is consistent with this basic contention of Heisenberg's gedanken experiment on γ-ray microscope, Margenau[5] from a critical analysis has pointed out that the non-statistical notion of Heisenberg's uncertainty relation in a simultaneous measurement of canonically conjugate dynamical variables, pertaining to a single particle, is not amenable to any meaningful interpretation within the framework of standard QM. There is no doubt that Heisenberg's notion of uncertainty played an important role in the initial stages of the development of QM and even now it offers in many cases a quick glimpse of some peculiar quantum results. The use of a mixture of classical and quantum concepts is its real strength, because one can get some picture of what may be happening to the individual 'particles' and in most cases it offers a semi-classical explanation for some quantum phenomena. As such the so called 'Heisenberg's uncertainty principle (HUP)' is a part of the semi-classical 'old quantum theory'. Home and Sengupta[6] have discussed a reformulation of the γ-ray microscope thought experiment congruous with the statistical interpretation of uncertainty relation.

The quantum mechanical uncertainty relation in the form of an inequality relating the respective standard deviations (note 4) of position and momentum observables for arbitrary statefunctions was first derived by Kennard[7] in Heisenberg's immediate succession, using the position–momentum commutation relation and invoking the Cauchy–Schwartz inequality (for inner products), also in 1927. Taking a cue from Born's probabilistic interpretation of quantum mechanical statefunction ψ, Weyl[8] in 1928 showed independently that the non-commutativity of position and momentum operators is actually a statement about the variances leading to the inequality relation.

$$\sigma_q \sigma_p \geq \frac{\hbar}{2}; \quad \text{where } \sigma_q^2 = \langle q^2 \rangle - \langle q \rangle^2, \text{ etc.} \tag{1}$$

In the general formulation of QM, any pair of non-commuting operators is subject to similar lower bounds for the uncertainty products and Kennard's inequality was generalized in 1929 by Robertson[9] for all observables (represented by self-adjoint operators $\hat{A}$ and $\hat{B}$) in the form

$$\sigma_A \sigma_B \geq \frac{|\langle [\hat{A}, \hat{B}] \rangle|}{2}, \quad \text{where } \sigma_A^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2, \text{ etc.} \tag{2}$$

Schrödinger[10] soon realized an important modification to Robertson's inequality by adding a new term for quantum states for which the covariance of the two operators is non-zero (note 5) and derived a more general inequality in the form

$$\sigma_A^2 \sigma_B^2 \geq \frac{\langle [\hat{A}, \hat{B}] \rangle^2 + (\langle [\hat{A}, \hat{B}]_+ \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle)^2}{4}, \tag{3}$$

where $[\hat{A}, \hat{B}]_+ = \hat{A}\hat{B} + \hat{B}\hat{A}$ is the anticommutator of $\hat{A}$ and $\hat{B}$.

This result is usually referred to as Robertson–Schrödinger inequality. For quantum states with zero covariance of $\hat{A}$ and $\hat{B}$, this relation reduces to Robertson's inequality. In this sense it is more general and can be applied to any two observables of a large class of states of quantum systems. Quantum states with non-zero covariance include coherent and squeezed states.

Apart from the most common position and linear momentum relation (given by the Kennard's inequality), the relation between (i) any two orthogonal components of the total angular momentum operator of a system and (ii) the number of electrons in a superconductor and the phase of its Ginzburg–Landau order parameter[11] can be deduced. The Robertson–Schrödinger uncertainty relation may be generalized[12] in a straightforward way to describe mixed states as

$$\sigma_A^2 \sigma_B^2 \geq \frac{(\text{tr } \hat{\rho}[\hat{A}, \hat{B}])^2 + (\text{tr } \hat{\rho}[\hat{A}, \hat{B}]_+ - \text{tr}(\hat{\rho}\hat{A})\text{tr}(\hat{\rho}\hat{B}))^2}{4}, \tag{4}$$

where $\hat{\rho}$ is the density operator representing the mixed state.

From a proposed estimation of purity/mixedness of a system using quantum networks[13], Mal *et al.*[14] have recently claimed to establish a connection between the generalized Robertson–Schrödinger uncertainty relation and the mixedness of quantum states.

In his seminal paper of 1927, Heisenberg apparently gave intuitive formulation of three manifestations of uncertainty relations: (i) the uncertainty relations for the widths/spreads representing the intrinsic fluctuations of the distributions of two conjugate dynamical variables in a quantum state; (ii) for the inaccuracy of a measurement of one of the quantities and the ensuing disturbance in the distribution of the other and (iii) for the inaccuracies of any joint measurements of these quantities, representing the general invasiveness of measurements in quantum systems. However, it was only the relation for state preparations (type (i)) that was made precise soon afterwards. For example, the Kennard's/Robertson's inequality states that the quantum fluctuations of both position and momentum cannot be suppressed simultaneously below a certain limit. The inequality expressing the quantum mechanical uncertainty relation actually represents the intrinsic fluctuations in quantum states and follows directly from the formalism of QM.

The uncertainty relations of the first form (i) actually reveal a fundamental property of quantum systems, and is not a statement about the observation produced effect. According to Mermin[15], the uncertainty relations as deduced from within the framework of QM, prohibit the possibility of preparing an ensemble of systems in which all the members have sharply defined values for the canonically conjugate variables. This minimal interpretation states that it is not possible to prepare pure ensembles in which all systems have the same values for these quantities, or ensembles in which the spreads are smaller than allowed by quantum theory (note 6). In the following sub-sections, we discuss subsequent developments of this form of uncertainty relations in terms of other measures of 'uncertainty' and the formulation of the time–energy uncertainty – its special character and interpretation.

The minimal interpretation does not address the question whether one can make simultaneous/joint measurements of position and momentum. As a matter of fact, one can show that the standard formalism of QM does not allow a description of joint measurements[16]. In his intuitive formulations, Heisenberg presented types (ii) and (iii) of the uncertainty relations 'as a statement of empirical fact' of the inevitable and uncontrollable disturbance of a quantum system by the measuring apparatus and 'it took several decades until the conceptual tools required for a rigorous formulation of the two measurement-related uncertainty relations had become available'[17]. We shall discuss these developments leading to the formulation of 'universally valid uncertainty relations', in the following sections together with the recent studies highlighting subtle interconnections of the uncertainty relations with other important quantum features like nonlocality, entanglement, steering, separability, etc. Finally, we discuss the experimental validation of uncertainty relations and the lively debate concerning Popper's criticisms vis-à-vis the various experimental studies.

The uncertainty relations in the context of harmonic analysis

In Schrödinger representation, the appearance of uncertainty relations has a simple explanation in the expressions of the quantum mechanical statefunctions in the two conjugate bases (position and momentum), which are Fourier transforms of one another. Hardy first provided a rigorous mathematical statement of Wiener's observation regarding Weyls formulation of uncertainty relation that a quantum state and its Fourier transform cannot both be well-localized[18]. Hardy's study shows that both a function and its Fourier transform cannot be simultaneously 'very rapidly decreasing'. The trade-off between the 'spreads' of a function (f) and its Fourier transform ($\bar{f}$) leads to an uncertainty relation. With the observation that a Gaussian function is a minimizer for eq. (1), Hardy[19] suggested to measure the spread/localization of f and $\bar{f}$ with respect to a Gaussian distribution. Hardy's theorem has been subsequently generalized and extended for other exponential functions, replacing the Gaussian[20].

Yet another important aspect is revealed in the fact (Paley–Wiener theorem) that it is not possible for both f and $\bar{f}$ to be compactly supported (unless of course they vanish entirely). If f is compactly supported, then $\bar{f}$ is an entire function. Due to the in-built Fourier correlation between the coordinate and momentum representations, a more stringent condition than the uncertainty relation that the quantum mechanical state function cannot have a compact support in both position and momentum representation is imposed[21]. It is impossible in the statefunction description for a system to have compact distribution functions for both position and momentum density, i.e. to know both position and momentum with finite errors. Since the statefunction representation provides the complete description of a system in QM, the problem is that we cannot demand finite error in both position and momentum measurements. For a classical particle it is always possible to know both position and momentum with finite errors, i.e. both position and momentum density functions must be compact. This is impossible in the statefunction description because band limited functions cannot be spatially compact[21]. Not only macroscopic objects, states of microscopic particles for which this type of measurement is possible (α, β particle tracks) also seem to defy ψ-function description.

A variety of inequalities can be found in Fourier analysis providing mathematical formulations for uncertainty relations[22]. It is pertinent to note here that Bohr's derivation of uncertainty relations relied on Fourier analysis and not on the commutation relations. For Bohr, the indeterminacy relations are essentially an expression of wave–particle duality in QM, which he further expounded as the 'complementarity Principle' in his Como lecture[23]. However, the 'wave–particle trade-off' is now expressed in terms of an inequality, known as Englert–Greenberger duality or simply wave–particle duality relation[24]. The inequality derived involving suitably defined visibility V of the interference (pattern) and distinguishability D between the two interfering state-functions is given by

$$D^2 + V^2 \leq 1, \tag{5}$$

the equality holds for pure quantum states. Also, it is now generally accepted that entanglement (nonlocal quantum correlation) is essential for a proper explanation of the disappearance of the interference pattern due to welcher-weg/which-state detection ('complementarity') in interference experiments, which has traditionally been inadequately explained in terms of position–momentum uncertainty relation. The quantum interference effect is much more general than the classical wave-like interference in ordinary position space and the make-shift uncertainty argument is not even applicable for all types of which-state-interference complementarity[25]. One can derive an uncertainty-like inequality from the duality relation eq. (5), so both are related and the duality inequality appears stronger. We will again come to this point later on.

Finally, it may be pointed out that the two derivations of the fundamental uncertainty relations (based on Fourier analysis and on the commutation relations) pertaining to the state preparations are equivalent as far as the relationship between two conjugate observables (dynamical variables) are concerned. But this is not so for time and energy, since most physical systems do not have a time operator. We discuss the formulation and interpretation of the time–energy uncertainty relation in the following section.

The time–energy uncertainty relation

Since time at which the particle (system) has a given state is not an operator belonging to the particle, it is a parameter describing the evolution of the system. The time–energy uncertainty relation (TEUR), therefore, does not follow directly from the Robertson–Schrödinger inequality. However, using the classical analogy that no wave of finite duration can have a definite frequency, a formal deduction of TEUR follows from the time–frequency Fourier analysis. A relation involving the 'external time' may be interpreted in terms of the duration of a perturbation or preparation process and the corresponding uncertainty in energy of the state of the system developed.

At the same time it should be noted that besides the external coordinates of the space–time background, there may be internal temporal variables of clocks connected with a specific physical system, just like the internal spatial variables – the position variables of the particles of the system[2]. These are characteristic, internal/intrinsic dynamical time variables which obey the equations of motion for the system. In general, the evolution of an observable of a system provides a definition for an internal time variable of the physical system and also an approach to formulate the time–energy uncertainty relation.

It is well known from the spectroscopic analysis that decaying excited states having a finite lifetime do not have a definite energy – fast-decaying states have broad, while slow-decaying states have narrow linewidths. As the linewidth broadens, the determination of the energy of a state becomes less accurate (note 7). Actually the atomic de-excitation experiments highlight a different type of TEUR for the 'internal' time variable. We shall discuss presently the formulation of the time–energy uncertainty, developed in 1945 by Mandelshtam and Tamm[26]. However, when Heisenberg declared that the TEUR is 'a direct consequence of the familiar equations

$$Et - tE = -i\hbar \quad \text{or} \quad Jw - wJ = -i\hbar, \tag{6}$$

where J is the action variable and w is the angle variable', it creates much confusion than clarification. First, how familiar the given equations are and how one should interpret them in view of the relational word 'or'? Only in an early publication prior to Heisenberg's paper, Dirac[27] arrived at a similar equation (the first one of the two) by turning the (external) time parameter t into an internal variable by the procedure described there. In the same paper by Heisenberg the subsequent comment, 'As time can be treated as a parameter (or a c-number) when there are no time-dependent external forces', made things worse since time-dependent forces did not enter in the discussion of eq. (6) either, and there, time was certainly not a c-number. It appears that Heisenberg's view on time and specially the introduction of his 'familiar' equations are quite confusing. In fact, Heisenberg posited a time operator conjugate to the Hamiltonian without addressing Pauli's theorem[28], which inhibits the concept of a universal time operator valid for all systems.

From his analysis of the Stern–Gerlach experiment, Heisenberg finally arrived at the relation $E_1 t_1 \sim h$, where E_1 is the imprecision in the energy measurement by the apparatus and t_1 the time during which the atom is under the influence of the deflecting field and concludes that a precise determination of energy can only be obtained at the cost of a corresponding uncertainty in the time. This leads to an early wrong interpretation which claims that measuring the energy of a quantum system to an accuracy requires an extended observational time interval. This was ridiculed in Lev Landau's joke: 'To violate the time–energy uncertainty relation all I have to do is measure the energy very precisely and then look at my watch!'[3]. Aharonov and Bohm[29] have shown that the time in the uncertainty relation is the time-duration during which the system remains unperturbed and not the time during which the experimental equipment is turned on. The present-day QM capitulates that all observables can be measured with arbitrary accuracy in an arbitrarily short (external) time and the energy is no exception to this[30]. It is quite evident that Heisenberg's semi-classical arguments and perceptions regarding TEUR are not quite part of the quantum mechanical description and when pushed too far, logical contradictions will appear.

Following Mandelshtam and Tamm[26], we now trace the TEUR for an 'internal' time to the non-commutation of the Hamiltonian $(\hat{H})$ and an arbitrary quantum mechanical observable, say $\hat{A}$ of any physical system. The product of the standard deviations of $\hat{A}$ and $\hat{H}$ is given according eq. (2) as

$$\sigma_A \sigma_E \geq \frac{|\langle[\hat{A},\hat{H}]\rangle|}{2}, \quad \text{where } \sigma_E^2 = \langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2, \tag{7}$$

together with the quantum mechanical equation of motion of $\hat{A}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle = \hbar^{-1}\,|\langle[\hat{A},\hat{H}]\rangle|, \tag{8}$$

we have

$$\sigma_A \Big/ \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle \geq \hbar/(2\sigma_E). \tag{9}$$

Now, for a particular quantum mechanical state, the r.h.s. is fixed. So, for any arbitrary $\hat{A}$, the l.h.s. has a lower bound and the dimension of time. If we define the average time taken for the expectation value of some observable $\hat{A}$ of the system to change by its standard deviation as a measure of the average lifetime of a state and denote it with σ_τ, then we have a time–energy uncertainty-like relation of the form

$$\sigma_\tau \sigma_E \geq \frac{\hbar}{2}, \tag{10}$$

where σ_E, the spread in the energy value, may be considered as the spectral linewidth of the state.

Thus the time–energy uncertainty relation may appear in two forms – one concerns the time of preparation of a given state and the other the lifetime of composite systems. Furthermore, Busch[30] has classified a variety of valid measures for the 'dynamical time' and the related TEUR formulations and also described a formal representation of the 'event time' observables in terms of positive operator valued measures (POVM) (note 8; ref. 31) over the relevant time domain. A TEUR in this case, though not universally valid, holds good in specific cases of Hamiltonian and time domain.

In popular textbooks on QM, the TEUR is often interpreted to suggest a temporary violation of the conservation of energy with the implication that energy can be 'borrowed' from the universe as long as it is 'returned' within a very short time. This is not a valid theoretical proposition and commenting on this Griffiths[32] writes: 'Nowhere does quantum mechanics license violation of energy conservation, and certainly no such authorization entered into the derivation' (of the TEUR). Clearly, this misconception is based on the false axiom that the energy of the universe is an exactly known parameter at all times. It is argued that when events transpire at shorter time intervals, there is a greater uncertainty in the energy of these events. Therefore it is not that the conservation of energy is violated when quantum field theory uses temporary electron–positron pairs in its calculations, but that the energy of quantum systems is not known with enough precision to limit various possibilities.

In the case of decay of an unstable subatomic particle, the intermediate state is represented by virtual particles. The Casimir effect is a manifestation of the reality of virtual particles. Time–energy uncertainty relation offers an understanding to the creation of virtual particles and the range of the exchange interactions mediated by them since the (virtual) particles involved are created and exist only for a brief while during the exchange process.

Other measures of uncertainty – the entropic uncertainty relations

The uncertainty relation, defined in terms of variances/standard deviations of the corresponding conjugate dynamical variables is meaningful only if both the standard deviations are finite. It can be shown that apart from the well-known Cauchy wavepackets, there are a number of other interesting statefunctions whose spatial standard deviations diverge. Also, there are states having large standard deviations of position, but are actually a superposition of very narrow bumps. For such states, the momentum uncertainties may be much larger than those obtained from Robertson's inequality. Again, if $\hat{A}$ and $\hat{B}$ are two observables of a finite n-level system, then σ_A and σ_B are always finite and vanish in eigenstates of $\hat{A}$ and $\hat{B}$ respectively. Therefore, no uncertainty relation of the form eq. (2) except the trivial one: $\sigma_A \sigma_B \geq 0$, is possible at all.

In fact, the propriety of variance as a measure of uncertainty is quite limited. Also, there is nothing special about the second moment, we could as well use the $2n$th moment, where n is an integer. Then the error/uncertainty in position q would be $(\langle q^{2n}\rangle - \langle q\rangle^{2n})^{1/2n}$. In fact, the 'kurtosis' defined in terms of the fourth moment captures the 'peakedness' of a distribution. In harmonic analysis, as discussed earlier, it may be noted that Hardy has considered the rate of decay of a function at infinity as the natural measure of its concentration. Benedicks[33], on the other hand, uses Lebesgue measure to define concentration/localization in terms of smallness of support and showed that for a non-zero function, it is impossible for both the function and its Fourier transform to have finite Lebesgue measure.

While formulating the many-worlds interpretation of QM in 1957, Everett[34] arrived at a new inequality based on information theory. He concluded that the new relation is stronger than the variance-based relation, since it implies the former but is not implied by the former. From the point of view of information theory, uncertainty is just the lack of information and can be measured in exactly the same manner as information is measured. As a measure of uncertainty – the deficiency in the information or negative information, the information entropy is to be used with a reversal of sign. Following Everett[34] and Hirschman[35], the so called 'entropic' uncertainty relations have been proposed[36–38] involving the Shannon entropy[39] as the measure of uncertainty, which is much closer to the spirit of quantum von Neumann entropy (note 9; ref. 40).

The Shannon entropy is a special class of family of Rényi entropies[41] in the limit of order 1. For a general probability distribution $P = (P_1,..., P_i,..., P_N)$, $P_i \geq 0$, $\sum_i P_i = 1$, on a set of N possible outcomes, the Shannon entropy is defined by the following formula closely resembling the definition of the physical entropy

$$\mathcal{H} = -\sum_i P_i \ln P_i. \tag{11}$$

With normalized Gaussian distribution, this entropy amounts to the variance (for a fixed variance the Gaussian maximizes the entropy) producing the Shannon-type inequality for 'entropic' uncertainty relation. If the spatial standard deviation of a statefunction is infinite, then it is appropriate to take for the position and momentum uncertainties, their respective information entropies and use the corresponding entropic uncertainty relation. In this alternative form of entropic uncertainty relation, the sum of entropies of the considered non-commuting observables is employed.

In a finite dimensional Hilbert space, applying this notion to the probability distributions of two physical observables $\hat{A}$ and $\hat{B}$, having complete sets of non-degenerate (normalized) eigenfunctions $|a_i\rangle$s and $|b_j\rangle$s respectively, in the state $|\psi\rangle$, Deutsch[37] has derived the relation

$$\mathcal{H}(A) + \mathcal{H}(B) \geq -2 \ln\left(\frac{1+c}{2}\right), \tag{12}$$

where $\mathcal{H}$(A) denotes the Shannon entropy of the probability distribution of the outcomes of $\hat{A}$ measurement, etc. and $c = \max_{i,j} |\langle a_i|b_j\rangle|$ denotes the overlap of these measurements. Partovi[38] soon generalized the relation to the case when the spectra of $\hat{A}$, $\hat{B}$ have degeneracies which reduce to Deutsch's inequality in the non-degenerate limit.

Following an idea for improvement from Kraus[42], Maassen and Uffink[43] have suggested a stronger inequality for any two observables with nondegenerate spectra in a finite-dimensional Hilbert space, in the form:

$$\mathcal{H}(A) + \mathcal{H}(B) \geq -2 \ln c. \tag{13}$$

Subsequently, Krishna and Parthasarathy[44] have generalized the result of Maassen–Uffink for any pair of observables in a finite-dimensional Hilbert space, which reduces to eq. (13) when the observables have non-degenerate spectra. It may be pointed out that the Krishna–Parthasarathy inequality is stronger than the Partovi bound. Maassen and Uffink have also studied and obtained a general set of inequalities and indicated extension of these relations for infinite dimensional Hilbert space.

The advantage of these relations over variance-based Robertson relation is that the right-hand side is always independent of the state $|\psi\rangle$. Entropic uncertainty relations have attained increasing prominence in recent studies. The usefulness of various generalizations and extensions of the entropic uncertainty relations in quantum theory, their direct connections to the observed phenomena and some open problems are discussed in recent reviews[45]. In another interesting development, Berta *et al.*[46] have shown that the lower bound on the uncertainties of the measurement outcomes depends on the correlations between the observed system and an observer who possesses a quantum memory (which can store information about a quantum state better than any classical memory). Using a smoother version of the concept of min- and max-entropy (note 10), the authors have derived a lower bound on the uncertainties, which depends on the amount of entanglement between the particle and the quantum memory and discussed in detail the application of their result to witnessing entanglement and to quantum key distribution. The effectiveness of quantum memory in reducing quantum uncertainty has been demonstrated in two recent experiments using pure[47] and mixed[48] entangled states respectively. Pati *et al.*[49] have subsequently derived an uncertainty relation by incorporating an additional term that depends on the quantum discord and the classical correlations of the joint state of the observed system and the quantum memory that tightens the lower bound of Berta's relation. Recently, Pramanik *et al.*[50] have obtained further reduction to the uncertainty lower bound in the presence of quantum memory using the fine-grained entropic uncertainty relation developed by Oppenheim and Wehner[51] (the 'fine-grained entropy' takes into account all details of the probability distribution, whereas the coarse-grained entropy is a smeared-out averaged version over the entire phase space). In the process they have claimed to attain the ultimate or optimal lower bound for the derived uncertainty relation, which is independent of the measurement settings.

We shall revert to the discussions of other important revelations by the entropic uncertainty relation in the subsequent sections.

Origins of quantum uncertainty: how quantum are the uncertainty relations?

It has already been pointed out that Heisenberg in his celebrated '1927 paper' envisaged, though vaguely, three manifestations of the uncertainty relations. Whilst conceptually distinct, these three kinds of uncertainty relations are now shown to be formally closely related.

The notion of uncertainty is inherent in classical wave mechanics. It is well known that, whereas a plane progressive wave spreads over the entire space, a localized wavepacket is spread out in momentum space. Similarly, to make a wave pulse shorter – limited in time, more and more frequency components are to be added implying an 'uncertainty' in frequency of the pulse. It also appears in the signal processing, a generalization and refinement of Fourier analysis when the signal frequency characteristics vary with time. The basic result of this analysis, developed in concert with QM in the 1930s and 1940s and referred to as the Heisenberg–Gabor limit or simply the Gabor limit[52], which limits the simultaneous time-frequency resolution, is that a function cannot be both time-limited and band-limited. Waves and particles are clearly denizens of the classical world – two distinct categories of physical entities, having entirely different mathematical descriptions and there is no mix-up (note 11; ref. 53). But in QM this basic difference in representations of physical entities disappears and the typical wave–particle duality sets in. The fundamental quantum mechanical uncertainty manifested by Robertson inequality is already shown to be derivable from both the non-commutativity of conjugate observables and harmonic analysis. We have also noted earlier that this type of indeterminacy relations are expression of wave–particle duality inherent in all quantum systems as reiterated by Bohr. It is also pertinent to note that probabilistic theories with inherent degree of incompatibility (of two observables) greater than that of QM are possible. Based on a concept of a 'global measure', Busch *et al.*[54] have introduced the idea of comparing probabilistic theories with respect to the non-classical feature of incompatibility.

The two measurement-related uncertainty relations, on the other hand, appear due to the operator formulation of the dynamical variables (observables) and serious attempts of a rigorous formulation of these relations are made only recently. First, we note that the quantum measurements are generically invasive and measuremental disturbance cannot be calculated accurately. But non-invasive measurements are possible in classical mechanics (CM) and even for invasive measurements, the measuremental disturbance may be calculated and accounted for accurately (note 12). The putative uncertainty described in Heisenberg's thought experiment, is one arising from the measuremental error and concomitant induced disturbance on the conjugate observable.

The original formulation of Robertson's inequality implies a limitation on the state preparation and prediction from the past. It refers to independent measurements of the observables $\hat{A}$ and $\hat{B}$ on distinct but identically prepared systems of an ensemble in a state represented by the density operator, say, $\hat{\rho}$ and not to the limitations on measuremental observations as envisaged in type (ii) and (iii) uncertainties. An entropic formulation of the uncertainty relations for the systematic error and disturbance of a quantum system incurred in sequential/successive measurements (type ii) was first introduced by Srinivas[55]. But finally, it was Ozawa, who took the initiative for a holistic approach to the problem for the derivation of an universally valid uncertainty relation by considering both the uncertainty due to the measurement disturbance and the uncertainty already accrued in a quantum state (during the preparation) before the measurement. Starting with some reasonable definitions for the error ε_A of a measurement on an observable $\hat{A}$ and the disturbance η_B on the observable $\hat{B}$ caused by the measurement, Ozawa[56] has 'reconstructed' a reciprocal relation between ε_A and η_B in the spirit of Heisenberg's error–disturbance relation given by

$$\varepsilon_A \eta_B \geq \frac{|\langle[\hat{A},\hat{B}]\rangle|}{2}. \tag{14}$$

On the other hand, various aspects of the problem of joint measurement of two noncommuting observables were first elucidated by Arthurs and Kelly, and Arthurs and Goodman[57]. From a realistic theoretical model for approximate joint measurement of noncommuting observables $\hat{A}$ and $\hat{B}$, the related uncertainty relation type (iii) may be written as

$$\varepsilon_A \varepsilon_B \geq \frac{|\langle[\hat{A},\hat{B}]\rangle|}{2}, \tag{15}$$

under the joint unbiased measurement condition requiring that the experimental mean values of the outcome a of the $\hat{A}$-measurement and the outcome b of the $\hat{B}$-measurement should coincide with the theoretical mean values of observables $\hat{A}$ and $\hat{B}$ respectively, on any input state ψ. Two noncommuting observables $\hat{A}$ and $\hat{B}$ of a system can be approximately measured on the system states ψ by the measurements of two commuting Hermitian operators (theoretical approximants) $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ on the extended Hilbert space of the system plus apparatus states. Ozawa[56] has introduced a standard definition for the inaccuracies in an approximate joint measurement of $\hat{A}$ and $\hat{B}$ in terms of the statistical deviations

$$\varepsilon_A^2 = \mathrm{tr}[\hat{\rho}(\hat{A}-\hat{\mathcal{A}})^2];\ \varepsilon_B^2 = \mathrm{tr}[\hat{\rho}(\hat{B}-\hat{\mathcal{B}})^2], \tag{16}$$

where tr denotes the trace.

Ozawa has further pointed out a logical relationship between the noise–disturbance and the joint measurement inaccuracies. He argued that if in a sequential measurements of two observables the measurement of $\hat{A}$ using any apparatus is immediately followed by a measurement of $\hat{B}$ using a noiseless second measuring apparatus, the entire process is equivalent to an approximate joint measurement of $\hat{A}$ and $\hat{B}$ on the input state of the apparatus for $\hat{A}$-measurement. The noise/inaccuracy of $\hat{B}$-measurement (approximate joint) is then equal to the disturbance caused by the measuring apparatus of $\hat{A}$ to the observables $\hat{B}$ and we have the relation: $\eta_B = \varepsilon_B$. Although there has been some initial apprehension regarding the experimental accessibility of these quantities, two independent methods[58,59] have nevertheless been proposed and implemented recently in several experimental works.

The naively taken noise–disturbance relation given by eq. (14) lacks a rigorous foundation and using the above definitions, Ozawa has concluded that the validity of this relation cannot be reduced to the uncertainty relation for joint measurements (eq. (15)). He has also discussed the criterion for the validity of this relation and observed that it fails to hold in general. This assertion has been vindicated recently in a series of experimental works[60–63] which we will discuss in some detail in a following section. Briefly, according to Ozawa, the common assertion that any measurement of an observable $\hat{A}$ with inaccuracy ε_A must invariably lead to a minimum disturbance to the conjugate observable $\hat{B}$ of the order consistent with eq. (14), is incorrect. Eventually, the recent experimental claims regarding the general failure of the naive error-disturbance and error–error relations, have sparked a stimulating debate concerning the proper mathematical definition of 'error' and 'disturbance'. Busch *et al.*[64] in a recent communication have contradicted Ozawa's contention and proposed a new definition for the error–disturbance quantities. They have taken the maximized difference between the respective probability distributions before and after the measurement, over all localized states, as the measure for both error and disturbance to obtain the 'Heisenberg-type inequality' (eq. (14)). However, it may be noted that this inequality does not hold for any given state since the error and disturbance are defined for different localized states. These quantities appear to describe the noise and disturbing power of the measuring device, whereas eq. (16) (Ozawa's definition) quantifies the error and disturbance in a measurement for the same quantum state. We shall return to this discussion presently.

Starting from Robertson's relation and using a relevant triangle inequality, Ozawa finally developed, without imposing any constraint on the statefunctions, except for the positive metric Hilbert space with the natural commutator algebra, a single general relation

$$\varepsilon_A\eta_B + \varepsilon_A\sigma_B + \sigma_A\eta_B \geq \frac{|\langle[\hat{A},\hat{B}]\rangle|}{2}, \tag{17}$$

the so-called Ozawa's measurement–disturbance relation (MDR) or the universally valid (error–disturbance) uncertainty relation (UVUR). It may be noted that the intrinsic fluctuation, the unsharpness due to the measuremental errors and the concomitant induced disturbance of the conjugate observables are all, in the final expression, ultimately related to the noncommutativity of the corresponding operators.

It should be noted that a relation similar but inequivalent to Ozawa's relation was derived subsequently by Hall[65], which involves the standard deviations (SDs) of two commuting approximate observables rather than the SDs of the actual noncommuting observables and depends more on the particular choice of $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$. Very recently, inspired by the works of Ozawa and Hall, a stronger and tighter relationship for the error and disturbance as defined by Ozawa has been derived by Branciard[66]. Acknowledging the importance and validity of Ozawa's relation, Branciard has argued that it is not optimal because the three independent terms of this relation cannot be saturated simultaneously in general. The (sub)optimality of Ozawa's relation is improved using a more general geometric inequality to provide a modified universally valid MDR. It has the property that it may be satisfied while the 'Heisenberg-type' expression is violated and also Ozawa's relation follows directly from Branciard's relation. However, Hall's inequality, which involves more quantities, does not follow from Branciard's relation and on optimization can provide an even stronger relation.

In a subsequent development, Fujikawa[67] has suggested to combine the Ozawa's inequality (eq. (17)) with Robertson's relation (eq. (2)) to exploit certain freedom in the identification with the form (eq. (14)) of the error–disturbance uncertainty relation. The resulting inequality, may be written in a simple form with two factors on the left-hand side as

$$\bar{\varepsilon}_A\bar{\eta}_B \geq |\langle[\hat{A},\hat{B}]\rangle|, \text{ where } \bar{\varepsilon}_A = \varepsilon_A + \sigma_A \text{ and } \bar{\eta}_B = \eta_B + \sigma_B. \tag{18}$$

It is claimed that this relation can be proved with the same mathematical rigour of Kennard–Robertson relations, without making the assumptions such as unbiased measurement, with a clear physical meanings for the two factors as the total error/inaccuracy in the measured values of the observable $\hat{A}$ and the resulting total disturbance in the conjugate observable $\hat{B}$. Fujikawa and co-workers[68] have used the Heisenberg picture for the time development of quantum mechanical operators in incorporating the effects of the measurement interaction to show that all the measurement and disturbance relations are secondary consequences of Robertson's relation. They have further demonstrated that the Arthurs–Kelly–Goodman error–error or the naive error–disturbance relations can be obtained using the assumptions of unbiased measurement and unbiased disturbance. These simplified uncertainty relations are thus only conditionally valid. On the other hand, despite the claim of the vindication of 'Heisenberg-type inequalities' by Busch *et al.*, Ozawa has cited a class of solvable models which escape the Busch–Lahti–Werner assertion.

Fujikawa *et al.* have also argued that the notion of precise measurement is incompatible with the unbiased measurement assumption. The unbiased measurement and disturbance lead to unbounded disturbance for bounded operators such as spin variables. On the other hand, increasing error does not always lead to decreasing disturbance and vice versa in spin measurements. Such a reciprocal behaviour occurs only in certain cases and along certain directions. Important practical issues appear in quantum estimation theory, high-precision measurements using quantum theory and exploiting quantum entanglement (quantum metrology) and in quantum cryptography (to decide whether a communication channel has been 'sniffed' or not), which are strictly adhered to the unbiased measurement and disturbance assumption and use the error–disturbance relation. Ozawa's and the recent experimental studies have raised the possibility of furtive quantum measurements, barely disturbing a system and also the question – how far is the quantum encryption really secure? All these works suggest that the mathematical formulation, used to decide whether a quantum channel is secure or not, might have to be revised.

In this connection it may be mentioned that recently Li *et al.*[69] have proposed a new scheme to express the uncertainty principle in the form of inequality of the bipartite correlation functions for a given multipartite state, which provides an experimentally feasible and model-independent way to verify various uncertainty and measurement disturbance relations.

In view of the absence of a full-fledged theory of time measurements, it may be relevant to point out here that an appropriate time–energy UVUR is yet to be developed. Also, the usual entropic uncertainty relations describe informational contents and do not refer to the interaction of quantum measurements. They rather represent a generalization of Robertson's relation (eq. (2)) for probability distributions for which the variance is an inadequate measure of uncertainty. Srinivas[55] has initiated the formulation of a 'different class of entropic uncertainty relation' and derived an optimal bound on the uncertainties when two or more observables are sequentially measured on the same system. This optimal bound is shown to be greater than or equal to the bounds on the uncertainties for the observables which are traditionally derived for the putative measurements on distinct but identically prepared systems of an ensemble. Finally, it would be interesting if the universally valid uncertainty relation of Ozawa (eq. (16)) also has an entropic counterpart and promising research in this direction is in progress as evident from the recently archived preprints[70,71].

Classical and quantum uncertainty – correlation and entanglement

Consequently, Luo[72] has discussed the subtilities of the description of uncertainty and correlation of quantum mechanical mixed states and pointed out that even for those pure states, where the description of uncertainty in terms of variance may be adequate, two kinds of contributions, one quantum and the other classical, appear when the states are mixed. It is, therefore, desirable to split the 'fluctuation' (variance) into two parts, one quantum and the other classical. For a quantum state represented by the density operator $\hat{\rho}$, the variance of an observable $\hat{A}$ in $\hat{\rho}$ may be alternatively expressed as

$$\sigma_A^2 = \mathrm{tr}\, \hat{\rho}\hat{A}^2 - (\mathrm{tr}\, \hat{\rho}\hat{A})^2.$$

Lou has identified the Wigner–Yanase skew information (introduced by Wigner and Yanase[73]) to quantify the information content of an observable not commuting with $\hat{A}$ (conjugate or skew to $\hat{A}$) in the state $\hat{\rho}$, defined by

$$q_A^2 = -\frac{\mathrm{tr}[\hat{\rho}^{\frac{1}{2}}, \hat{A}]^2}{2},$$

where [,] represents the commutator bracket, as the appropriate measure of quantum uncertainty, which is identically zero in a state of classical mixture of eigenstates of $\hat{A}$ and the difference: $c_A^2 = \sigma_A^2 - q_A^2$, gives the classical uncertainty which is zero for a pure state.

The author has also worked out a similar decomposition of the conventional covariance into quantum and classical parts leading to the corresponding new measures for quantum and classical uncertainties and correlations through a polarization procedure. He has finally derived 'a new uncertainty relation in purely quantum terms' and quantified 'nonlocal quantum correlation' or entanglement to provide a new perspective for understanding uncertainty, correlation and entanglement for mixed states.

The notion of uncertainty is generally perceived as to express a limitation of operational possibilities in the quantum regime. However, its full content, as it continually emerging, both creates new insights for practical propositions and relates with other concepts of QM. The interconnections of the uncertainty relations with other quantum features such as entanglement, nonlocality, etc. have been revealed through a number of recent studies. In order to demonstrate the EPR correlations for dynamical variables having a continuous spectrum, Reid[74] first proposed an 'eminently practical' method by setting a criterion – the violation of an inequality involving products of the inferred variances of noncommuting observables.

For states with positive Wigner function, measurements on a pair of noncommuting continuous variables do not exhibit violation of a Bell-type inequality, since outcomes of such measurements obey the predictions of a local hidden variable theory. While discussing the problem of detection of EPR correlations in these cases, Reid has derived an experimentally testable criterion in the form of an inequality – the above mentioned 'inferred' Heisenberg uncertainty relation. These EPR correlations are, therefore, manifested not by violations of Bell inequalities, but by demonstrating a sufficient correlation between results of measurements performed at two spatially separated locations, through the violation of 'EPR inequalities'. In 1992, Ou *et al.*[75] reported the first experimental demonstration of the violation of Reid's 'EPR inequality', using the parametric oscillator method, suggested by Reid and Drummond[76]. Subsequently, Zhang *et al.*[77] detected EPR correlations between the intense output fields of the parametric oscillator above threshold frequency. It is noteworthy that a violation of an inferred Heisenberg uncertainty relation for measurements in the context of EPR correlations was first predicted by Reid and was subsequently verified experimentally.

The steering interpretation of this type of work provides another window for exploration (note 13; refs 78 and 79). Steering is a form of quantum nonlocality occupying an intermediate position between Bell nonlocality and entanglement. Cavalcanti *et al.*[80] have developed a more general 'EPR-steering criterion' which can rederive the previous 'EPR criterion', and studied its efficacy in detecting steering in some classes of quantum states. However, for arbitrary non-Gaussian states, the inequality based on variances (of noncommuting observables) is not very effective and one needs to use the 'entropic steering inequality' developed by Walborn *et al.*[81]. In terms of this general criterion, Chowdhury *et al.*[82] have examined the steering property in several classes of important non-Gaussian entangled states.

Oppenheim and Wehner[51] have recently shown, using their 'fine-grained entropy', that the presence of uncertainty via steering, directly limits the maximally achievable nonlocality. They have shown that for bipartite systems with unbiased measurement settings, the link between uncertainty and nonlocality holds for all physical theories. The formalism developed with 'fine-grained entropy' for bipartite systems and its subsequent extension to the case of tripartite systems[83] can be used to discriminate on the basis of the strength of nonlocality among classical, quantum and 'superquantum' correlations. More recently, Dey *et al.*[84] have shown that the fine-grained entropic uncertainty relation is still able to distinguish classical, quantum and superquantum correlations (involving two or three parties), but for biased settings corresponding to certain ranges of the biasing parameters only. This study helps locate the region of advantage for different theories, to explore the extent of nonlocality that can be captured by regulating the bias parameters and the possibility of a generalization to multiparty nonlocal retrieval games. Moreover, the strength of nonlocality can, in turn, determine the strength of uncertainty in measurements. Hari Dass *et al.*[85] have recently shown that entanglement puts a bound on the product of uncertainties of non-commuting observables, for certain class of systems and states. A complementary result subsequently reported by Tomamichel and Hanggi[86] claims that in order to achieve a certain nonlocality, at least some specific amount of uncertainty is necessary.

On the other hand, 'separability' of a given state as an alternative to the criterion for entanglement is also found to be expressible in terms of uncertainty relations. Separability is the property that distinguishes statistical ensembles having a classical description from those that need a quantum description. Formulations of 'separability criteria' with both the variance-based[87] and entropy-based[88] uncertainty are available in the literature.

It is already pointed out that in quantum interference experiments, the mutual exclusiveness of the two complementary properties, the precise which-state information and the appearance of sharp interference pattern, is enforced by the entanglement of the interfering statefunction branches with orthogonal detector states. Any tentative explanation using uncertainty relations provides only naive semi-classical arguments. Nevertheless, it appears that though 'uncertainty' and 'complementarity' are two independent notions, in some cases they are inextricably related.

Real experiments corroborating and confronting the uncertainty relation

The first high-precision experimental test for the uncertainty relations came about only in 1969 from Shull's[89] single-slit neutron diffraction experiment. Later in the 1980s followed the neutron interferometric experiments by Kaiser *et al.*[90] and Klein *et al.*[91]. More recently, in 2002, Nairz *et al.*[92] have reported a demonstration of uncertainty relation by measuring the increase in momentum spread of the molecular beam of fullerene (C_{70})

molecules after passage through a narrow slit with a variable width.

In these diffraction and interferometric experiments, typical measures used for the width of the spatial state-function are the slit width and slit separation respectively, to confirm the familiar version of uncertainty relations (the variance based state preparation relations). The mathematical modelling of these experiments make use of a Gaussian approximation for the central peak and relating the half widths to standard deviations, the authors confirm the correct lower bound for the uncertainty product. Here, the momentum uncertainty is inferred semi-classically from the measured position distribution of the particles at the detection screen using far-field approximation. With this observation, Busch *et al.*[17] have pointed out that these analyses of the experimental data do not provide model-independent, direct confirmation of the uncertainty relation and discussed its possible model-independent validation.

The reports on experimental verification of measurement–disturbance uncertainty relations have started appearing in the literature only recently. The neutron-optical experiment by Erhart *et al.*[60] records the error of a spin component measurement as well as the disturbance caused on another spin component. It is claimed that their experimental results demonstrated the varied quantum uncertainty sources and even when either the source of error or disturbance is held to nearly zero, the other remains finite. Both the error and disturbance obey the new measurement–disturbance relation[56], derived using the theory of general quantum measurements, but violate putative Heisenberg's error-disturbance relation (eq. (14)), valid only under specific circumstances, in a wide range of an experimental parameter. This is the first experiment which distinguishes different sources of quantum uncertainty and there has been some discussion on the significance of this experiment.

However, Rozema *et al.*[61] have pointed out that, rather than directly characterizing the effects of an individual measurement, this work checked the consistency of Ozawa's theory by carrying out a set of measurements from which the disturbance could be inferred through tomographic means. In contrast, following Lund and Wiseman's[59] proposal of using weak measurement (note 14; refs 93 and 94) to experimentally characterize a system before and after it interacts with a measurement apparatus, they claimed to have directly measured its precision and the disturbance. From the experimental result they have also contended that while Ozawa's MDR formulation remains valid for all the experimentally tested parameters, the simplistic error–disturbance relation (eq. (14)) always falls below the experimentally measured bound. It is finally concluded that, 'although correct for uncertainties in states, the form of Heisenberg's precision limit is incorrect if naively applied to measurement' and that the experiment 'highlights an important fundamental difference between uncertainties in states and the limitations of measurement in quantum mechanics'.

Of the two independent methods, used so far to test the validity of different error–disturbance uncertainty relations experimentally, the three-state method[58] requires the preparation of multiple input states and is simpler to implement for a single qubit systems[60,62,95,96]. The weak-measurement method[59], on the other hand, more closely resembles the classical approach for measuring errors and is more feasible in general cases[61,97]. Ringbauer *et al.*[63] have observed that both methods were defined under ideal conditions which are unattainable in practice and extended the respective estimation procedures to account for experimental imperfections. They have tested Branciard's new relations by performing approximate joint measurements of incompatible polarization observables on single photons using both the three-state and the weak-measurement method. Their results show that while Ozawa's relation (eq. (17)) is universally valid and indeed satisfied by their data (but not saturated, as it is not tight), the relations, eqs (14) and (15) (Arthurs–Kelly–Goodman) are only conditionally valid under some restrictive assumptions.

The time–energy uncertainty relation, on the other hand, has been verified in various atomic, nuclear and particle decay experiments. In these experiments, the 'uncertainty' in time should be interpreted as the duration of the state since its creation. If the state is unstable with lifetime τ, then as the external time $t \to \infty$, the uncertainty in the frequency can be expected to settle to its 'natural' value of $\hbar/\tau$. This was observed by Wu *et al.*[98] and others utilizing the Mössbauer effect in γ-rays. Time indeterminacy was also demonstrated in quantum beats experiment in neutron interferometry by Badurek *et al.*[99].

Popper's criticisms

We conclude this critique with a brief discussion of Popper's experimental proposal aimed at falsifying uncertainty relations which generated a great deal of interest and controversy. In his original proposal of 1934, Popper[100] conceived a joint measurement scheme on entangled particle pairs to test indeterminacy in QM, which he referred to as 'statistical scatter relations' and disagreed with its application to individual systems. However, following some crucial objections from von Weizsäcker and Einstein, Popper accepted their criticisms and withdrew the proposition. Later, he suggested (footnote on p. 15 of Popper[101]) that his example may nevertheless have inspired to conceive the famous EPR thought experiment and subsequently he returned to the subject with new arguments, which was finally published in 1982 (ref. 101). In the new version, entangled pair of quantum particles is emitted in opposite directions from a source to pass through slits on either side, one narrow and the other wider. While predicting diffraction patterns of appropriate slit width on each side, Popper argued that according to the Copenhagen interpretation it should be, on the contrary, same diffraction pattern by the particles on either side.

Initially, Popper's proposed experiment was considered by many as a crucial test of QM, although some crucial flaws in the proposal were subsequently pointed out by Sudbery[102], Collet and Loudon[103] and Redhead[104], after scrutinizing the prospect of an actual realization of the experiment. Sudbery has pointed out that while the momentum spread being infinite for a perfectly correlated state, no further spread would be produced by localizing one particle of the correlated pair(s). Collet and Loudon argued that for the particle pairs originating from the source with zero total momentum, the blurring introduced by the uncertainty in the position of the source would wash out the Popper effect. From an analysis of Popper's experiment with a broad source Redhead, on the other hand, concluded that it could not yield the effect that Popper was seeking.

Circumventing the objections of Collet and Loudon through innovative arrangements, Popper's experiment was realized in 1999 by Kim and Shih[105] using a spontaneous parametric down-conversion (SPDC) photon source. They did not observe an extra spread in the momentum of particle 2 due to particle 1 passing through a narrow slit. In fact, the observed momentum spread was narrower than that contained in the original beam. This observation seemed to imply that Popper was right. However, Kim and Shih asserted that this result does not constitute a violation of the uncertainty principle and observed 'Popper and EPR were correct in the prediction of the physical outcomes of their experiments. However, Popper and EPR made the same error by applying the results of two-particle physics to the explanation of the behavior of an individual particle. The two-particle entangled state is not the state of two individual particles. Our experimental result is emphatically NOT a violation of the uncertainty principle which governs the behavior of an individual quantum'.

Contrary to this assertion, from a theoretical analysis of this experiment using the path-integral formulation, Sancho[106] found a similar narrowing in the momentum spread of particle 2, as was observed by Kim and Shih. Clearly, this calculation indicated that the experimental result of Kim–Shih is in agreement with the quantum mechanical prediction. On the other hand, Short[107] argued that the experimental outcome in this case is a consequence of the imperfect imaging process which leads to image blurring and there is indeed no violation of the uncertainty relation.

Recently, Qureshi[108] has analysed Popper's proposal and showed that the mere presence of slit A in the path of particle 1 does not lead to a reduction of the statefunction and any spread in the momentum of particle 2, even if the source is improved to give a better correlation. For if such an experiment could increase the momentum spread of that present in the initial state, it would lead to the possibility of faster than light communication[109]. Commenting on the outcome of Kim and Shih experiment, Qureshi maintained that: 'Popper may have been right in saying that there would be no spread, but for the wrong reasons' and asserted that Popper's prediction could easily have been tested in this experiment by gradually narrowing slit *A*, and observing the corresponding diffraction pattern behind slit *B*. Using the formalism of QM, Krips[110] has predicted that narrowing slit *A* would lead to momentum spread increasing at slit B. Qureshi has further pointed that this effect has actually been demonstrated experimentally in the so-called two-particle ghost diffraction experiment[111]. This experiment was not carried out with the purpose of testing Popper's ideas, but ended up giving a conclusive result about Popper's test. In this experiment, two entangled photons travel in different directions. Photon 1 goes through a slit, but there is no slit in the path of photon 2. However, photon 2, if detected in coincidence with a fixed detector behind the slit detecting photon 1, shows a diffraction pattern. The width of the diffraction pattern for photon 2 increases when the slit in the path of photon 1 is narrowed. Thus, increase in the precision of knowledge about photon 2, by detecting photon 1 behind the slit, leads to increase in the scatter of photon 2. In both the experiments of Kim and Shih, and Strekalov *et al.*, formalism of QM has been vindicated against the challenge posed by Popper's proposal.

One may argue here that Popper's original contention regarding the violation of the uncertainty relation using an entangled pair has ultimately been vindicated in the recent experiments[47,48] showing the reduction of uncertainty in the presence of quantum memory (aside from the violation of an 'inferred' Heisenberg-like uncertainty relation in the demonstration of EPR correlations[75,77] or as in the case of observed violations[60,61] of the naively-taken Heisenberg-like uncertainty relation). In spite of some basic flaws[102–104] in the original analysis, Popper's intuitive recognition of the problem shows great insight. However, it should be pointed out in the same breath that his challenge to the foundation of QM has turned out to be misplaced.

A further interesting recent claim may be noted in this context. Using the results of Oppenheim and Wehner[51], Hanggi and Wehner[112] have argued in a recent work that violation of quantum mechanical uncertainty relation(s) leads to a thermodynamic cycle (quantum) with positive net work gain, meaning a breakdown of the second law of thermodynamics and is very unlikely.

Notes

1. In popular scientific literature the term Heisenberg's Uncertainty Principle (HUP) is most common. However, Heisenberg never seems to have endorsed the term 'principle' and presented the

relations as the result of a 'pure fact of experience'. His favourite terminology was 'inaccuracy relations' (Ungenauigkeitsrelationen) or 'indeterminacy relations' (Unbestimmtheitsrelationen).

2. A state that only exists for a short time cannot have a definite energy. In order to have a definite energy, the frequency of the state must be defined accurately, and this requires the state to hang around for many cycles, the reciprocal of the required accuracy[3].
3. 'While position operators are commonplace, no time operators occur in ordinary quantum mechanics (QM). In view of relativity theory this seems problematic. Most present-day textbooks emphasize that space and time play fundamentally different roles in quantum mechanics. ... time poses no fundamental problem for quantum mechanics. If by space and time one understands the coordinates of a given space and time background, none of these coordinates are operators in quantum mechanics. If, on the other hand, one thinks of position and time as dynamical variables (obeying equations of motion) of a specific physical system situated in space–time, the representation of such variables by quantum mechanical operators is possible'[2].
4. Variance or standard deviation (the positive square root of variance) provides a simple measure of the fluctuation/spread around the mean value for many distributions.
5. For two random variables one can define three second moments: the variances of each observable and their covariance. In Kennard's/Robertson's relation only the two variances are involved.
6. Although interpretations of QM, in which each system has a definite value for its position and momentum are still viable, this is not to say that they are without strange features of their own and they do not imply a return to classical physics.
7. The same linewidth effect also makes it difficult to measure the rest mass of fast decaying particles in particle physics. The faster the particle decays, the less certain is its mass.
8. In situations where the outcomes are non-discrete and the measurements determine probability measures on a non-discrete outcome space, the associated operators have a general representation in positive operator valued measures (POVMs) – an element of which can be defined using the so-called Kraus operator[31]. POVMs may be seen as a completion of the notion of observable within the Hilbert space framework of QM when included with the traditional concept of observables in the form of projection valued measures (PVMs). A projective measurement on a large system acts on the subsystems in ways that cannot be described by a projective measurement on the subsystem alone. POVMs are recognized to provide appropriate description of the effects on the subsystems produced by a projective measurement on the whole system. Here, a rough analogy may be drawn with the role of density matrices in the description of part of a larger system which is in a pure state.
9. In quantum statistical mechanics, von Neumann entropy[40] is the extension of classical entropy concepts to the field of QM. For a quantum-mechanical system described by a density matrix ρ, the von Neumann entropy is $\mathcal{S} = -\mathrm{tr}(\rho \ln \rho)$, where tr denotes the trace. If ρ is written in terms of the eigenvectors $|1\rangle, ...|i\rangle$, ... etc. as $\rho = \sum_i \eta_i\, |i\rangle\, \langle i|$, then the von Neumann entropy is $\mathcal{S} = -\sum_i \eta_i \ln \eta_i$. In this form, $\mathcal{S}$ can be seen to be related to the Shannon entropy $\mathcal{H}$.
10. The name min/max-entropy stems from the fact that they are the smallest and largest entropy measures in the family of Rényi entropies (of order $\pm\infty$) respectively. In this sense, the min-entropy is the strongest way to measure the information content of a discrete random variable. In particular, the min-entropy is never larger than the Shannon entropy. According to the information-theoretic interpretation, the min-entropy represents the maximum achievable quantum correlation and the max-entropy gives the decoupling accuracy.
11. In classical wave propagation, Fourier analysis reveals uncertainty relations in the form of a reciprocal relationship between the widths of the spatial/temporal wave pattern and the wave number/frequency distributions. On the other hand, in classical statistical physics, relations akin to the quantum uncertainty relations in terms of the product of the root mean square (rms) fluctuations of a thermodynamic extensive variable with its conjugate intensive variable, representing stability equations of equilibrium thermodynamics, may be obtained[53]. These statistical relations as derived from the distribution functions, however, differ by a factor of 1/2 from the quantum uncertainty relations which are derived from the probability amplitudes (statefunctions).
12. For example, one can measure velocity in classical mechanics (CM) by stopping the body and measuring the work done in the process.
13. The term steering was introduced by Schrödinger[78] in 1935 for the fact that entanglement would seem to allow an experimenter to remotely steer the state of a distant system as in the Einstein–Podolsky–Rosen (EPR) argument. Einstein called this 'spooky action at a distance'. Steering has recently been rigorously formulated as a quantum information task, opening up to new experimental tests[79].
14. Weak measurement is a technique to measure the average value of a quantum observable $\hat{A}$ without appreciably affecting the state of the system being measured[93,94]. Weak measurements differ from normal (strong or 'von Neumann') measurements in two ways. A strong measurement of an observable $\hat{A}$ of a quantum system in an arbitrary state S yields an eigenvalue of $\hat{A}$. If the measurement is repeated many times (starting each time with the system in state S), one obtains a sequence of eigenvalues of the observable. The different eigenvalues appear with probabilities determined by the state S and the average of a large number of repeated measurement gives the expectation value $\langle\hat{A}\rangle$ of the observable in the state S. In a strong measurement, the initial pure state S is also changed/projected to an eigenstate of $\hat{A}$, i.e. the measured state is substantially changed unless it happened to be close to that eigenstate. By contrast, a weak measurement only yields a sequence of numbers which average to $\langle\hat{A}\rangle$. For example, a strong measurement of the spin of a spin-1/2 particle must yield spin 1/2 or –1/2, but a particular weak measurement could yield spin 100 (ref. 93), while a subsequent weak measurement value on an identical system might be –28. Therefore, a single weak measurement gives no information, except yielding correct expectation value. A weak measurement, as mentioned above, does not substantially change the initial state.

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