

# Graph theoretical modelling and analysis of fragile honey bee pollination network

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**Due to environmental stress honey bees frequently make foraging shifts. Encounter with any non-flowering agent during such shifts leads to colony collapse. To assess the impact of non-flowering agents in the pollination network, we have modelled the pollination network as polligraphs using graph theoretical tools. Based on the outcome of the model we have extracted a few new network parameters and characterized the stability of the pollination network.**

**Keywords:** Honey bees, non-flowering agent, polligraph, pollination network.

## Impact of urbanization

THE functional interaction of pollinators, especially insect pollinators, and their significance in most terrestrial ecosystems in stabilizing wild plants and crops have been well documented<sup>1-3</sup>. As honey bees predominantly manage to enhance agricultural production globally, Klein *et al.*<sup>4</sup> have pointed out that the decline in insect pollinators, particularly honey bees, will have serious impact on diverse plant industries. The sudden disappearance of the worker bee population from a single bee colony followed by rapid collapse and death of the colony has been reported and several suspected causes, including environmental stresses, malnutrition, unknown pathogens, mites and pesticides are currently being studied<sup>5</sup>. Several researchers have reviewed the causes for this decline and derived multiple driver concept for the loss of honey bees<sup>6-8</sup>.

Several causes such as pathogenic and parasitic infection<sup>9,10</sup>, environmental pollution mainly due to pesticides<sup>11</sup>, bioinvasion<sup>12,13</sup> and climate change<sup>14,15</sup> have been attributed to the loss of honey bees. But a few studies have focused their attention towards the effects of urbanization on pollinators<sup>16-18</sup>. The urban environment is constantly exposed to intense human activities<sup>19</sup> and the process of urbanization generates great amounts of waste materials<sup>20,21</sup>. As a consequence, pollinators are affected by lack of resources<sup>22</sup> and suitable habitats<sup>23</sup>. Starvation

and poor foraging conditions also operate as one of the losses<sup>24</sup>. This in turn urges the pollinators to shift to non-flower agents such as disposable paper cups with residual sugars thrown out of coffee shops and juice centres (Figure 1) to overcome the nutritional stress<sup>25</sup>. Such a foraging shift is not uncommon in honey bees and is due to short-term memory towards profitable food sources<sup>26</sup>. The shifting of honey bees to non-flower agents will upset the mutualistic relationship between honey bees and flowering plants, leading to crop loss. The impact of this change on ecosystem stability needs to be addressed since honey bees are potential pollinators<sup>27</sup> and their decline is directly linked with agricultural productivity. Here we introduce a mathematical tool, viz. graph theory widely used in analysing networks to study the pollination net in which flowers and non-flowers are nodes, and paths of honey bees are links to assess the disturbance in the pollination network. In recent years, there has been a surge of interest in the analysis of networks as models of complex systems. Graph theoretic tools have been widely used in analysing social and economic networks, information networks, technological networks and biological networks. In this article, we have proposed a graph theoretical model to stimulate pollination network and used graph theoretical tools to assess the disturbance in the pollination network.

## Problem description

Honey bees play a vital role in the process of pollination. Honey bees frequently switch over from flowering agents (Figure 2) to a few non-flowering agents<sup>25</sup>. As this shift acts as a death trap, it delinks the pollination network. Tracing the network and prediction of threat to fragile eco-balance will facilitate planning of conservation strategies.

## Model description

The pollination network described in Figure 3 has been illustrated as a directed graph (digraph) by considering flower and non-flower agents as vertices and the paths as arcs in Figure 4. The digraph has been designated as a

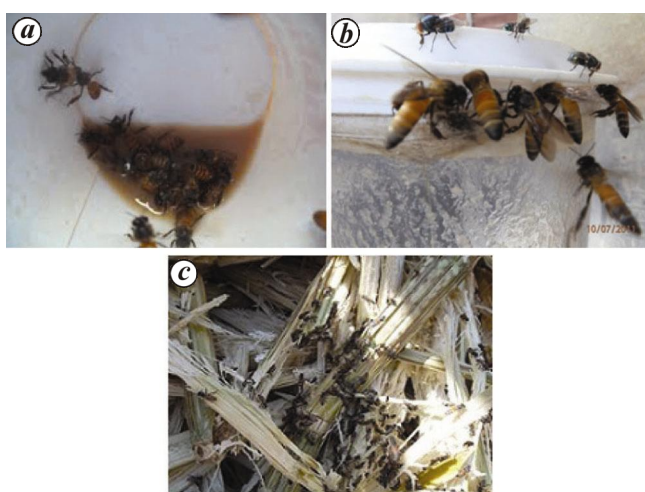
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polligraph  $P$  and characterized into partial stability, complete stability and unstable. Through graph theoretical properties of the polligraph, various parameters were evolved using Mathematica-9 and used for analysing the pollination network.

## Preliminaries

In this article, we follow the definitions and notations of graph theory<sup>28</sup>. A digraph  $D$  is an ordered pair  $(V(D), A(D))$  consisting of a set  $V = V(D)$  of vertices and a set  $A = A(D)$ , disjoint from  $V(D)$ , of arcs, together with an incidence function  $\psi_D$  that associates with each arc of  $D$  an ordered pair of (not necessarily distinct) vertices of  $D$ . If  $a$  is an arc and  $\psi_D = (u, v)$ , then  $a$  is said to join  $u$  to  $v$ ; we also say that  $u$  dominates  $v$ . The vertex  $u$  is the tail of  $a$  and the vertex  $v$  its head; they are the two ends of  $a$ . The number of vertices and arcs in  $D$  is called the order and size of  $D$  respectively. The reverse digraph  $D^-$  (which is sometimes called converse of  $D$ ) is obtained by reversing all the arcs of  $D$ . A digraph  $H$  is a subdigraph of a digraph  $D$ , if  $V(H) \subseteq V(D)$  and  $E(H) \subseteq E(D)$ . A spanning subgraph is a subgraph containing all the vertices of  $D$ . The indegree  $d^-(v)$  of a vertex  $v$  in  $D$  is the number of

arcs directed into  $v$  and the outdegree  $d^+(v)$  of a vertex  $v$  in  $D$  is the number of arcs going out of  $v$ . The minimum indegree and outdegree of  $D$  are denoted by  $\delta^-$  and  $\delta^+$  respectively; likewise, the maximum indegree and outdegree of  $D$  are denoted by  $\Delta^-$  and  $\Delta^+$  respectively. The total degree (or simply degree) of  $v$  is  $d(v) = d^+(v) + d^-(v)$ . If a digraph has the property that for each pair  $u, v$  of distinct nodes of  $D$ , at most one of  $(u, v)$  and  $(v, u)$  is an arc of  $D$ , then  $D$  is an oriented graph. A digraph  $D$  is symmetric if whenever  $(u, v)$  is an arc of  $D$ , then  $(v, u)$  is an arc of  $D$ . A complete graph which is oriented is a complete oriented graph. A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex, and a complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between every pair of vertices. A directed walk in  $D$  is a finite sequence  $W = v_0 a_1 v_1 a_2 v_2 \dots a_k v_k$ , where  $a_1, a_2, \dots, a_k$  are arcs and  $v_0, v_1, v_2, \dots, v_k$  are vertices of  $D$  such that  $v_{i-1}$  and  $v_i$  are respectively, the tail and the head of  $a_i$  for all  $i = 1, 2, \dots, k$ . A directed trail is a directed walk which does not repeat any arc. A directed path is a directed walk which does not repeat any vertex, except possibly the two ends. A directed cycle is a directed path in which the two vertices are the same. The length of a path is the number of arcs in it. A path of length  $l$  is denoted by  $l$ -path. A  $k$ -cycle is a cycle of length  $k$ . For a set  $S \subseteq V$ , define  $N^+(S) = \{w \notin S : \exists v \in S \exists (v, w) \in A\}$ ,  $N^-(S) = \{w \notin S : \exists v \in S \exists (w, v) \in A\}$ . A digraph  $D$  is strongly connected or strong, if  $D$  contains both  $u-v$  and  $v-u$  paths for every pair  $u, v$ , of distinct vertices of  $D$ . A digraph  $D$  is unilaterally connected or unilateral, if for each pair of vertices  $u$  and  $v$ , there is either a path from  $u$  to  $v$  or a path from  $v$  to  $u$  in  $D$ . A digraph  $D$  is weakly connected (or just connected) if the underlying graph of  $D$  is connected. A digraph  $D$  is said to be disconnected if it is not even weakly connected. The vertex connectivity  $\kappa(D)$  is defined as the minimum number of vertices whose removal from  $D$  results in a disconnected graph or in the trivial graph. The edge connectivity  $\lambda(D)$  is defined as the minimum number of edges whose removal from  $D$  results in a disconnected graph or in the trivial graph. A subgraph obtained by node deletions only is called an induced subgraph. If  $X$  is the set of nodes deleted, the resulting subgraph is denoted by  $D - X$ . A graphoidal cover of a digraph  $D$  is a collection  $\Psi$  of (not necessarily open) paths in  $D$  satisfying the following conditions: (i) Every path in  $\Psi$  has at least two vertices. (ii) Every vertex of  $D$  is an internal vertex of at most one path in  $\Psi$ . (iii) Every edge of  $D$  is in exactly one path in  $\Psi$ . The minimum cardinality of a graphoidal cover of  $D$  is called graphoidal covering number and is denoted by  $\eta(D)$ .



**Figure 1.** Non-flower agents chosen by honey bees. *a*, Tea cups; *b*, Juice centre cups; *c*, Sugarcane.



**Figure 2.** Flower agents.

**Theorem 1.** *A digraph  $D$  is strongly connected if and only if there does not exist  $S \subseteq V$ ;  $S \neq \emptyset$  such that  $N^+(S) = \emptyset$ .*

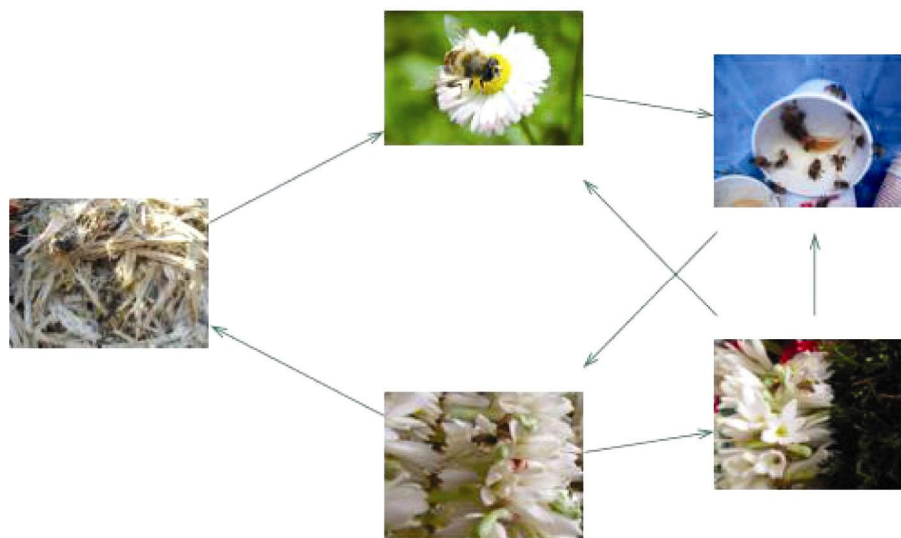


Figure 3. Pollination network.

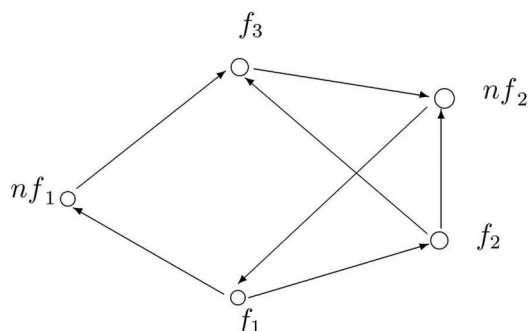


Figure 4. Digraph of pollination network in Figure 3.

**Definitions**

**Definition 1.** A polligraph  $P = (V, A)$  is a digraph with  $V = V_f \cup V_{nf}$ , where  $V_f$  a set of vertices associated with flower agents,  $V_{nf}$  a set of vertices associated with non-flower agents and  $A$  is the set of arcs. The order of  $P$  is  $n = n_1 + n_2$ , where  $|V_f| = n_1$  and  $|V_{nf}| = n_2$ . The size of  $P$  is  $m$ .

**Definition 2.** A polligraph is connected if for every pair of vertices there is at least one path between them; otherwise the polligraph is disconnected. Here onwards we mention connected polligraphs  $P$  otherwise mentioned.

**Definition 3.** A polligraph is completely stable if for every pair of flower agents  $x$  and  $y$ , there is a  $(x, y)$  path and  $(y, x)$  path in  $P$  such that there is no intermediate non-flower agent. A polligraph is partially stable if for every pair of flower agents  $x$  and  $y$ , there is either a  $(x, y)$  path or  $(y, x)$  path in  $P$  such that there is no intermediate non-flower agent.

*Example 1.* Examples are shown in Figures 5 and 6.

**Proposition 1.** Every completely stable polligraph is a partially stable polligraph. But the converse is not true. Figure 6 a is a partially stable polligraph but not a completely stable polligraph, since there is no path between the flower agents  $f_1$  and  $f_3$ .

Complete stability and partial stability can be characterized by the following theorems.

**Theorem 2.** A polligraph is said to be completely stable if and only if the subgraph induced by the flower agents is strong.

*Proof:*  $P$  is a completely stable polligraph if and only if for every pair of flower agents  $x$  and  $y$ , there is a  $(x, y)$  path and  $(y, x)$  path in  $P$  such that there is no intermediate non-flower agent, if and only if for every pair of flower agents  $x$  and  $y$ , there is a  $(x, y)$  path and  $(y, x)$  path in  $P - V_{nf}$ , if and only if  $P - V_{nf}$  is strongly connected.  $\square$

**Theorem 3.** A polligraph is said to be partially stable if and only if the subgraph induced by the flower agents is unilateral.

*Proof:*  $P$  is a partially stable polligraph if and only if for every pair of flower agents  $x$  and  $y$ , there is either a  $(x, y)$  path or  $(y, x)$  path in  $P$  such that there is no intermediate non-flower agent, if and only if for every pair of flower agents  $x$  and  $y$ , there is either a  $(x, y)$  path or  $(y, x)$  path in  $P - V_{nf}$ , if and only if  $P - V_{nf}$  is unilaterally connected.

**Proposition 2.** A complete oriented polligraph  $P$  of order  $n$  is completely stable if and only if there is no flower

agent having  $(n_1 - 1)$  indegree or outdegree in the subgraph induced by the flower agents.

*Proof:* Let  $P$  be a complete oriented polligraph on  $n$  nodes such that  $P$  is completely stable. Then  $d^+(v) \geq 1$ ,  $d^-(v) \geq 1 \forall v \in V_f$ . Hence the implies part. Conversely, assume the contrary. Then there exists a flower agent  $v$  having  $(n_1 - 1)$  indegree (outdegree) in  $P - V_{nf}$ . Thus there will be no arcs directed out of (into)  $v$ . Hence there exists no  $(v, x)((x, v))$  path  $\forall x \in V_f - v$ . Thus  $P$  is not completely stable. Hence the converse part.  $\square$

**Theorem 4** (ref. 29). *Let  $P$  be a completely stable polligraph such that  $P - V_{nf}$  is complete oriented. Then  $P - V_{nf}$  contains a directed  $k$ -cycle,  $\forall k = 3, 4, \dots, n_1$ .*

**Theorem 5** (ref. 29). *A polligraph  $P$  such that  $P - V_{nf}$  is complete oriented is completely stable if and only if  $P - V_{nf}$  has a directed  $n_1$ -cycle, where  $n_1 > 2$ .*

**Theorem 6** (ref. 29). *Let  $P$  be a completely stable polligraph such that  $P - V_{nf}$  is complete oriented and  $n_1 \geq 4$ . Then there exists a flower agent  $v_f$  of  $P$  such that  $P - v_f$  is a completely stable polligraph.*

**Definition 4.** A minimal set of flower agents  $F_{ct}$  of a completely stable polligraph  $P$  is said to be a complete pollination threat set if  $P - F_{ct}$  is not completely stable. A minimal set of flower agents  $F_{pt}$  of a partially stable polligraph  $P$  is said to be a partial pollination threat set if  $P - F_{pt}$  is not partially stable.

The minimum cardinality of a  $F_{ct}$  is called complete pollination threat number denoted by  $p_{ct}$ . The minimum

cardinality of a  $F_{pt}$  is called partial pollination threat number denoted by  $p_{pt}$ .

*Example 2.* Consider the polligraph in Figure 5 a,  $F_{ct} = F_{pt} = \{v\}$ , where  $v \in \{f_1, f_2, f_3\}$  is the complete (partial) pollination threat set. Hence  $p_{ct} = p_{pt} = 1$ .

**Definition 5.** A maximal set of arcs  $A_{cs}$  of  $P - V_{nf}$ , where  $P$  is a completely stable polligraph is said to be complete pollination strong set if  $P - A_{cs}$  is also completely stable. A maximal set of arcs  $A_{ps}$  of  $P - V_{nf}$ , where  $P$  is a partially stable polligraph is said to be partial pollination strong set if  $P - A_{ps}$  is also partially stable.

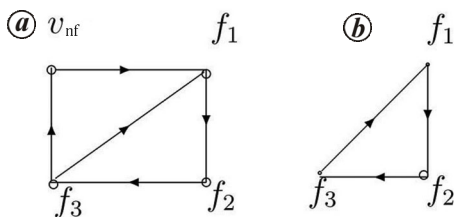
The maximum cardinality of a  $A_{cs}$  is called complete pollination strong number denoted by  $p_{cs}$ . The maximum cardinality of a  $A_{ps}$  is called partial pollination strong number denoted by  $p_{ps}$ .

*Example 3.* Consider the polligraph in Figure 7,  $A_{cs} = \{x, z\}$  and  $A_{ps} = \{x, y, z\}$ . Hence  $p_{cs} = 2$  and  $p_{ps} = 3$ .

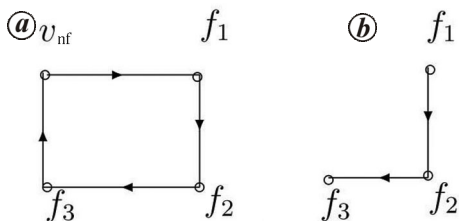
**Definition 6.** Pollination edge cut is a minimal set of arcs of  $P$  whose removal from  $P - V_{nf}$  leaves the resulting polligraph disconnected. Its cardinality is called polli edge cut number of  $P$  denoted by  $p_\alpha$ .

**Definition 7.** Pollination vertex cut is a minimal set of nodes of  $P$  whose removal from  $P - V_{nf}$  leaves the resulting polligraph disconnected. Its cardinality is called polli vertex cut number of  $P$  denoted by  $p_\beta$ .

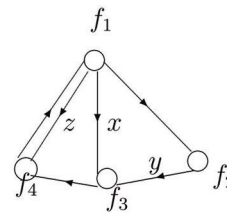
In general, a polligraph may be completely stable or partially stable or unstable. By completely stable we



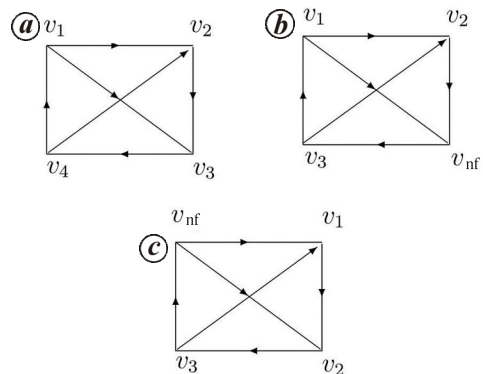
**Figure 5.** a, Completely stable polligraph. b,  $P - V_{nf}$  of (a), which is strong.



**Figure 6.** a, Partially stable polligraph. b,  $P - V_{nf}$  of (a), which is unilateral.



**Figure 7.**  $A_{cs} = \{x, z\}$  and  $A_{ps} = \{x, y, z\}$ .



**Figure 8.** (a) and (c) are completely stable polligraphs, but not (b).

mean the absence of disturbance in the natural pollination process by which the bees can move from one agent to another and vice versa. A completely/partially stable pollination network may destabilize by losing  $p_{ct}/p_{pt}$  number of flower agents. A flexible completely/partially stable pollination network will retain its stability even if  $p_{cs}/p_{ps}$  number of pathways is delinked. Delink of  $p_{\alpha}$  number of pathways and  $p_{\beta}$  number of flower agents will disconnect the system.

It is to be mentioned here that just the change of position of a non-flower agent affects stability of the polligraph. Thus, switch over of honey bees from pollination agents to a few non-pollination agents is a threat to the pollination network.

Assume that  $n_1 > n_2$  throughout this article.

**Theorem 7.** For any polligraph  $P$ ,  $p_{ct} \leq p_{pt}$ .

*Proof:* Since every completely stable polligraph is a partially stable polligraph, every partial pollination threat set is a complete pollination threat set. Hence by minimality,  $p_{ct} \leq p_{pt}$ .

**Theorem 8.** For any polligraph  $P$ ,  $p_{cs} \leq p_{ps}$ .

*Proof:* Since every completely stable polligraph is a partially stable polligraph, every complete pollination strong set is a partial pollination strong set. Hence by maximality,  $p_{cs} \leq p_{ps}$ .  $\square$

**Theorem 9** (ref. 30). For any polligraph  $P$ ,  $p_{\beta} \leq p_{\alpha} \leq \min \{ \delta^+, \delta^- \}$ .

**Proposition 3.** For every polligraph  $P$ ,  $1 \leq p_{ct} \leq n_1$ .

*Proof:* Since  $P$  is a completely stable polligraph,  $p_{ct} \neq 0$ , which implies  $p_{ct} \geq 1$ . Also,  $V_f$  is a complete pollination threat set to a polligraph  $P$  and hence by minimality,  $p_{ct} \leq n_1$ .  $\square$

**Proposition 4.** For every polligraph  $P$ ,  $1 \leq p_{pt} \leq n_1$ .

*Proof:* Since  $P$  is a partially stable polligraph,  $p_{pt} \neq 0$ , which implies  $p_{pt} \geq 1$ . Also,  $V_f$  is a partial pollination threat set to a polligraph  $P$  and hence by minimality,  $p_{pt} \leq n_1$ .  $\square$

**Proposition 5.** For every polligraph  $P$ ,  $0 \leq p_{cs} \leq n_1(n_1 - 2)$ .

*Proof:* Since  $p_{cs}$  is a non-negative integer,  $p_{cs} \geq 0$ . The number of edges in  $P - V_{nf}$  is at most  $n_1(n_1 - 1)$ . The number of edges in a strong digraph is at least  $m$ , where  $m$  is the total number of vertices. Therefore, for strong  $P - V_{nf}$ , the number of edges is at least  $n_1$ . Hence  $n_1(n_1 - 1) - p_{cs} \geq n_1$ , which implies  $p_{cs} \leq n_1(n_1 - 2)$ .  $\square$

**Proposition 6.** For every polligraph  $P$ ,  $0 \leq p_{ps} \leq n_1(n_1 - 2) + 1$ .

*Proof:* Since  $p_{ps}$  is a non-negative integer,  $p_{ps} \geq 0$ . The number of edges in  $P - V_{nf}$  is at most  $n_1(n_1 - 1)$ . The number of edges in a unilateral digraph is at least  $m - 1$ , where  $m$  is the total number of vertices. Therefore, for unilateral  $P - V_{nf}$ , the number of edges is at least  $n_1 - 1$ . Hence,  $n_1(n_1 - 1) - p_{ps} \geq n_1 - 1$ , which implies  $p_{ps} \leq n_1(n_1 - 2) + 1$ .  $\square$

**Proposition 7.** In a polligraph  $P$ , for every pair of flower agents there exists a  $l$ -path, for each  $1 \leq l \leq n_1 - 1$ , in which there are no intermediate non-flower agents, iff the subgraph induced by the flower agents is complete symmetric polligraph.

*Proof:* Let  $P - V_{nf}$  be a complete symmetric polligraph. Suppose there does not exist a path of length  $l$ , for some  $l$ ,  $1 \leq l \leq n_1 - 1$ , between a pair of flower agents in which there is no intermediate non-flower agents, then we get a contradiction to our assumption. Hence the implies part. Conversely, suppose  $P - V_{nf}$  is not a complete symmetric polligraph. Then there exists a flower agent whose indegree or outdegree is not equal to  $n_1 - 1$ , which implies there exists a positive integer  $l$ , for which there is no path between a pair of flower agents, which contradicts our assumption. Hence the converse part.  $\square$

Some special cases:

1. Let  $P$  be a complete symmetric polligraph. Then  $P$  is completely stable.  $p_{cs} = n_1(n_1 - 2)$ ;  $p_{ps} = p_{cs} + 1$ ,  $p_{\alpha} = d(v)$ .

Case (i)  $V_{nf} = \emptyset$ .

Both complete pollination threat set and partial pollination threat set do not exist.

Case (ii)  $V_{nf} \neq \emptyset$ .

$p_{ct} = p_{pt} = n_1$ .

2. For cycle polligraph of order  $n \geq 4$ .

Case (i)  $V_{nf} = \emptyset$ .

The given polligraph is completely stable (also partially stable).

$p_{cs} = 0$ ,  $p_{ps} = 1$ ,  $p_{ct} = 1$ ,  $p_{pt} = 2$ ,  $p_{\alpha} = p_{\beta} = 2$ .

Case (ii)  $V_{nf} \neq \emptyset$ .

(a) If non-flower agents are not adjacent to each other, then the given polligraph becomes unstable.

(b) If non-flower agents are adjacent to each other, then the given polligraph is partially stable but not completely stable. In this case,  $p_{ps} = 1$ ,  $p_{pt} = 1$ .

3. For the path polligraph of order  $n \geq 3$ .

Case (i)  $V_{nf} = \emptyset$ .

The given polligraph is partially stable but not completely stable.

$$p_{ps} = 0, p_{pt} = 1; p_{\alpha} = p_{\beta} = 1.$$

Case (ii)  $|V_{nf}| = 1$ .

(a) If the non-flower agent is the pendant vertex, then the given polligraph is partially stable but not completely stable.

$$p_{pt} = p_{\alpha} = p_{\beta} = 1, p_{ps} = 0.$$

(b) If the non-flower agent is not the pendant vertex, then the given polligraph is unstable.  $p_{\alpha} = p_{\beta} = 1$ .

### Strength of a polligraph

In pollination process, the existence of pollinators especially honey is mandatory<sup>31,32</sup>. During their shift from flowering to non-flowering agents, honey bees experience the threat of collapse. The magnitude of the fragility can be defined by calculating the strength  $S$  of the polligraph numerically.

**Definition 8.** Let  $P = (V, A)$  be a polligraph. Let  $f: V \rightarrow \{\delta, 1, 2\}$ ,  $0 \leq \delta < 1$  be a function defined by

$$f(v) = \begin{cases} \delta, & \text{if } v \in V_{nf}, \\ 2, & \text{if } v \in V_f \text{ and } d^+(v), d^-(v) \geq 1 \text{ in } P - V_{nf}, \\ 1 & \text{otherwise.} \end{cases}$$

$\delta$  is called the mortality rate. Consider the set  $L$  of all paths (open) of maximum length in  $P$ .

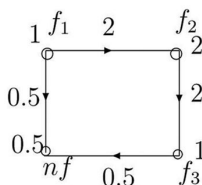
The strength  $S$  of a polligraph  $P$  is defined by

$$S(P) = \max_{e=(x,y) \in A(l)} \sum f(x) \cdot f(y) \quad \forall l \in L.$$

*Example 4.* The path of maximum length is  $f_1 f_2 f_3 n f$  (Figure 9). Strength  $S = 2 + 2 + 0.5 = 4.5$ .

**Theorem 10.** If  $P$  is a completely stable polligraph, then  $4n_1 - 4 \leq S < 5n_1 - 3$ .

*Proof:* Let  $L$  be the set of all paths (open) of maximum length in the polligraph  $P$ . Let  $l \in L$ . Since  $P$  is a completely stable polligraph,  $d^+(v), d^-(v) \geq 1$  in  $P - V_{nf}$ ,  $\forall v$  and there exists a path containing all flower agents. Therefore,  $f(v) = 2, \forall v \in V_f$  and the length of the path is  $n_1 - 1$ .



**Figure 9.** Maximum length path =  $f_1 f_2 f_3 n f$ .

$$\therefore S(P) = \max_{e=(x,y) \in A(l)} \sum f(x) \cdot f(y) \geq 4(n_1 - 1).$$

Also the path can be extended to non-flower agents. Let  $\delta_1, \delta_2, \dots, \delta_{n_2}$  be the mortality rate of non-flower agents  $nf_1, nf_2, \dots, nf_{n_2}$  respectively. Let  $\delta = \min\{\delta_i / i = 1, 2, \dots, n_2\}$ .

Then,

$$\begin{aligned} S(P) &= \max_{e=(x,y) \in A(l)} \sum f(x) \cdot f(y), \\ &\leq 4(n_1 - 1) + 2\delta + 2\delta + (n_2 - 2)\delta^2, \\ &< 4(n_1 - 1) + 4 + (n_2 - 2) (\because \delta < 1), \\ &\leq 4(n_1 - 1) + 4 + (n_1 - 1 - 2) (\because n_2 \leq n_1 - 1). \end{aligned}$$

Hence  $S(P) < 5n_1 - 3$ . □

**Proposition 8.** Polligraph  $P$  attains its maximum strength iff  $P$  is a completely stable polligraph.

*Proof:* Follows from Theorem 10. □

**Proposition 9.** For every polligraph  $P$ ,  $0 \leq S < 5n_1 - 3$ .

*Proof:* If non-flower agents exist between the flower agents, then the links receive the value 0. Therefore,  $S \geq 0$ . Upper bound follows from Theorem 10. □

**Theorem 11.** Let  $P = (V, A)$  be a polligraph. Either over all orientations of  $P$  or over all possible assignments of flower agents and non-flower agents to the vertices of  $P$ ,  $S(\text{CSP}) \geq S(\text{PSP}) \geq S(\text{UP})$ , where  $\text{CSP}$ ,  $\text{PSP}$  and  $\text{UP}$  denote completely stable, partially stable and unstable polligraphs respectively.

*Proof:* Let  $P$  be a polligraph. If  $P$  is completely stable then  $d^+(v), d^-(v) \geq 1$  in  $P - V_{nf}$ ,  $\forall v$ . This implies  $f(v) = 2, \forall v \in V_f$ . If  $P$  is not completely stable, then there exists  $S \subseteq V; S \neq \emptyset$  such that  $N^+(S) = \emptyset$  by Theorem 1. This implies that  $f(v) = 1 \quad \forall v \in S$ . So  $S(\text{CSP}) \geq S(\text{PSP})$  and  $S(\text{CSP}) > S(\text{UP})$ . Consider the  $l$ -path and  $m$ -path in partially stable and unstable polligraphs respectively, where  $l$  and  $m$  are maximum, then  $l \geq m$ . Thus  $S(\text{PSP}) \geq S(\text{UP})$ . Hence  $S(\text{CSP}) \geq S(\text{PSP}) \geq S(\text{UP})$ .

### Domination and graphoidal cover in polligraphs

#### Dominating sets

Domination in digraphs has been well studied. Domination number in polligraphs reveals that  $\gamma(P)$  flower agents dominate the pollination network based on the honey bee path.

**Definition 9.** A dominating set in a polligraph  $P$  is a set  $S$  of vertices of  $V$  such that every vertex  $u \in V - S$  has an



adjacent vertex  $v$  in  $S$  with  $v$  directed to  $u$ . The domination number of  $P$ , denoted by  $\gamma(P)$ , is the minimum cardinality of a dominating set in  $P$ .

**Theorem 12** (ref. 33). *For every polligraph of order  $n \geq 2$ , the following bounds are sharp:*

$$2 \leq \gamma(P) + \gamma(P^-) \leq \frac{4n}{3}, \quad 1 \leq \gamma(P)\gamma(P^-) \leq \frac{4n^2}{9}.$$

**Theorem 13** (ref. 34). *Let  $P$  be a weakly connected polligraph on  $n$  vertices. Then  $1 \leq \gamma(P) \leq n - 1$ .*

**Theorem 14** (ref. 35). *For a completely stable polligraph  $P$  on  $n$  vertices,  $\gamma(P - V_{nf}) \leq \lceil n_1/2 \rceil$ .*

**Theorem 15** (ref. 33). *For subpolligraphs  $P_1$  and  $P_2$  of a polligraph  $P$  with  $V(P_1) \cup V(P_2) = V(P)$ ,  $\gamma(P) \leq \gamma(P_1) + \gamma(P_2)$ .*

If all the dominating sets of a polligraph contain only flower agents, then the pollination network will sustain its stability. However, at least  $\gamma(P)$  number of flower agents should not become non-flower agents due to anthropogenic stress.

### Graphoidal cover

We defined graphoidal cover and graphoidal covering number for a polligraph to characterize the pollination network using the concept proposed by Acharya and Sampathkumar<sup>36</sup>.

**Definition 10.** A graphoidal cover of a polligraph  $P$  is a collection  $S$  of (not necessarily open) paths in  $P - V_{nf}$  satisfying the following conditions: (i) Every path in  $S$  has at least two vertices. (ii) Every flower agent of  $P$  is an internal vertex of at most one path in  $S$ . (iii) Every edge of  $P - V_{nf}$  is in exactly one path in  $S$ . Here path means directed path.

The minimum cardinality of a graphoidal cover of  $P$  is called graphoidal covering number and is denoted by  $\eta(P)$ .

**Theorem 16.** *Let  $P = (V, A)$  be a polligraph. Over all orientations of  $P$ ,  $\eta(CSP) \leq \eta(PSP) \leq \eta(UP)$ , where  $CSP$ ,  $PSP$  and  $UP$  denote completely stable, partially stable and unstable polligraphs respectively.*

*Proof:* It is enough to prove that, over all orientations of digraph  $D$ ,  $\eta(SCD) \leq \eta(UCD) \leq \eta(WCD)$ , where  $SCD$ ,  $UCD$  and  $WCD$  denote strongly connected, unilaterally connected and weakly connected digraphs respectively. Suppose over all orientations of a digraph  $D$ ,  $\eta(SCD) > \eta(UCD) > \eta(WCD)$ . Then there exist graphoidal

covers  $S_1, S_2, S_3$  such that  $\eta(SCD) = |S_1| > \eta(UCD) = |S_2| > \eta(WCD) = |S_3|$ , over all orientations of a digraph  $D$ . Since every strongly connected digraph is unilaterally connected and every unilaterally connected digraph is weakly connected<sup>32</sup>, there is a contradiction to our assumption. Hence the claim.  $\square$

**Proposition 10.** If  $\eta = 1$ , then the given polligraph  $P$  is completely stable or partially stable. In that case, the strength of  $P$  is maximum.

*Proof:*  $\eta = 1$  implies that there exists a path (open or closed) consisting of all flower agents. If the path is open (closed), then the polligraph  $P$  is partially (completely) stable.  $\square$

**Theorem 17** (ref. 36). *If  $P$  is a complete symmetric polligraph on  $n$  vertices with  $n_1 \geq 4$ , then  $P$  is completely stable and  $\eta = (n_1)^2 - 2n_1$ .*

The graphoidal covering number of a polligraph supports further analysis of the fragility of pollination network. A pollination network with the least (1) graphoidal covering number will be highly fragile and its increase will slowly reduce fragility.

### Software

Mathematica software version 9 was used to draw graphs, extract subgraphs and to enumerate the parameters with the following commands: `Graph[]`, `Subgraph[]`, `FindVertexCut[]`, `FindEdgeCut[]`, `EdgeConnectivity[]`, `VertexConnectivity[]`, `web = {}`, `shapes = {}`, `style = {}`, `Graph = [web, style]`.

### Conclusion

The biotic communities of an ecosystem can be characterized as networks of species (nodes) connected through interactions (links)<sup>37</sup>. In the present study we have analysed the pollination network with the values of the parameters  $p_{ct}, p_{pt}, p_{cs}, p_{ps}, p_{\alpha}, p_{\beta}, S, \gamma_F, \eta$  from the polligraphs (Figure 10). Table 1 reveals the characteristics of pollination network of the respective polligraphs.

The pollination networks corresponding to  $P_1, P_4$  and  $P_5$ , maintain sustained relationship between pollinators and pollinating agents and the threat to this complete stability is accomplished by the loss of a single flower agent. In accordance with Kearns *et al.*<sup>38</sup>, loss of oral resources is a key threat facing pollinating insects. Complete stability will be lost only if more than five pathways delinked in  $P_1$ , one and three pathways respectively, for  $P_4$  and  $P_5$ . Loss of one flower agent in  $P_1$  and  $P_3$  and two in  $P_4$  and  $P_5$  leads to the failure of partial stability in their pollination network. Disturbance in pollination network of  $P_1$ ,

$P_3$ ,  $P_4$  and  $P_5$  occurs only when more than six, three, two and four pathways disappear. Disappearance of  $P_\beta$  number of flower agents and  $P_\alpha$  number of honey bee pathways results in disconnection in pollination network. Analysis of strength reveals that  $P_5$  is the strongest and  $P_2$  is the weakest pollination network. The flower dominating number represents the number of dominant flower agents in the pollination network, i.e. more  $\gamma_F$  for less stable network against less  $\gamma_F$  for stable network. The graphoidal covering number will be maximum and minimum correspondingly for stable and unstable pollination networks. Theoretical and mathematical modelling studies carried out by Dunne *et al.*<sup>39</sup>, and Fortuna and Bascompte<sup>40</sup> have also indicated such a kind of delicacy

in mutualistic network due to the loss of nodes that are linked to a large number of other nodes.

As the enumeration of graphs for specific parameters is a huge class, characterizing such a class will be proving necessary and sufficient condition between the class of graphs and certain parameters, which is purely theoretical and will be another direction for research.

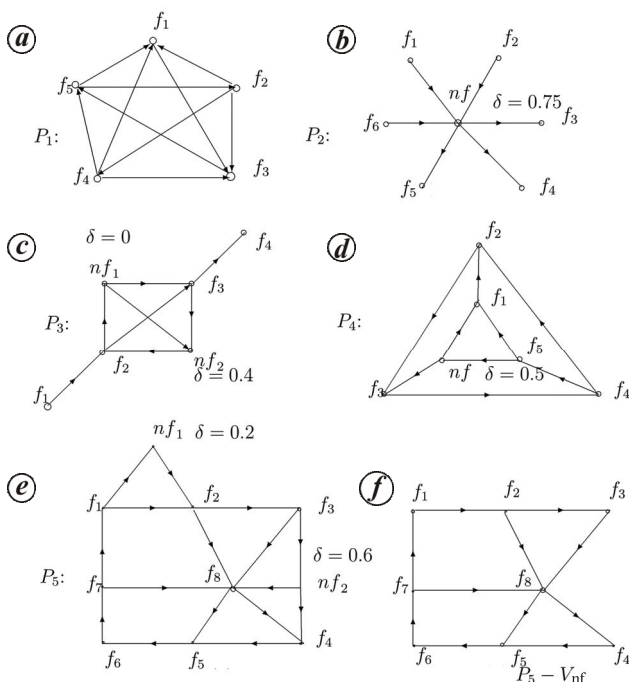


Figure 10. A few pollination networks.

Table 1. Characteristics of pollination network of the respective polligraphs

Polligraph →	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$
Completely stable	✓	–	–	✓	✓
Partially stable	✓	–	✓	✓	✓
Unstable	–	✓	–	–	–
$p_{ct}$	1	–	–	1	1
$p_{pt}$	1	–	1	2	2
$p_{cs}$	5	–	–	1	3
$p_{ps}$	6	–	3	2	4
$p_\alpha$	4	0	1	3	2
$p_\beta$	–	0	1	3	2
$S$	16	1.5	8	17	28
$\gamma$	2	4	3	2	4
$\eta$	5	–	1	2	3

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