R. P. Bambah: renowned mathematician, enthusiastic teacher and an able administrator

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Ram Prakash Bambah is an eminent mathematician, enthusiastic teacher, great researcher, well-known educationist and an able administrator. It is indeed rare to find such a combination of qualities of head and heart in an individual.

Bambah, son of Bhagat Ram Bambah and Lajwanti, was born on 30 September 1925 at Jammu. He studied at MB High School (Wazirabad) and K. C. Arya High School (Sialkot, Punjab, now in Pakistan). After passing his matriculation examination in 1939 with a top rank, he joined Government College, Lahore. His brilliance in mathematics soon impressed his class fellows who promptly nick-named him 'theta'. He stood first in the University in FA, BA, BA (Hons) and MA (Mathematics). In MA, he set an unbeatable record by scoring 600 marks out of 600. He was one year senior to Abdus Salam, who was successful in improving his records, except in the MA exam. Bambah had long-lasting friendship with Abdus Salam, who later won the Nobel Prize.

Bambah's teacher at Government College Lahore, the eminent number theorist Sarvadaman Chowla, was greatly impressed with him. In Bambah's own words: 'Professor Chowla was a real successor to Srinivasa Ramanujan, the greatest mathematician of Indian origin.' In 1946, Bambah decided to pursue research under the guidance of Chowla. At that time Punjab University had only two research scholarships on the whole. On the recommendation of Chowla, one was given to Bambah. He had an exciting time working with Chowla, who had an exceptional passion for mathematics. His earliest papers were sent by Chowla to G. H. Hardy, who had brought Ramanujan to Cambridge. Hardy edited and combined these papers to publish them as a single paper in the *Journal of the London Mathematical Society*.

Recognizing Bambah's extraordinary talent, D. S. Kothari, Head of the Department of Physics at Delhi University, strongly recommended him to the Vice Chancellor Sir Maurice Gwyer for a lecturership in the Department. Soon Bambah decided to leave Lahore due to the impending partition of the country and took up this challenging assignment on 1 February 1947. In 1948, he left for Cambridge with a prestigious scholarship – the 1851 Exhibition Scholarship (the Royal Exhibition of 1851 had created a fund to support research studies in Commonwealth countries; one scholarship was allotted to India). Bambah joined as a research student of L. J. Mordell, one of the great mathematicians of the century. Mordell suggested to Bambah a research problem in January 1949, which he was able to solve by April, to Mordell's great disbelief. Mordell had thought that the problem was too difficult to solve in such a short time. He looked at the details only after J. W. S. Cassels had confirmed that they seemed to be correct. Bambah was awarded a Ph D by the University of Cambridge in 1950. He spent the remaining one year of his scholarship working with Davenport at University College, London. During 1950–51, he wrote three papers in quick succession with Davenport, C. A. Rogers and K. F. Roth (who later got the Fields Medal, regarded as equivalent of the Nobel Prize). Bambah was elected to the fellowship of St John's College, Cambridge from 1952 to 1955.

Bambah returned to India in 1951. After a few months he was offered a research fellowship of Rs 500 per month for one year by the National Institute of Science of India (now Indian National Science Academy (INSA), New Delhi). In April 1952, Bambah was offered Readership in Panjab University (PU), with its Mathematics Department at Hoshiarpur. At the same time, he was offered a membership at the Institute of Advanced Studies at Princeton, USA. Soon after joining PU, the then Vice-Chancellor Dewan Anand Kumar gave Bambah leave for two years to enable him to avail of the opportunities in Cambridge and Princeton.

In 1954, while returning from England by ship, Bambah met Saudamini Parija who was returning after obtaining her MRCP. They got married on 23 February 1956 and were soon blessed with a daughter. In 1957–58, Bambah was invited as a visiting professor at Notre Dame University and was then selected as a professor in the Department of Mathematics at PU in 1958 at the young age of 33. The only other professor in the Department was Hans Raj Gupta. Both of them contributed substantially to the progress of the Department, which was recognized as a Centre for Advanced Study in Mathematics by UGC in 1962, a status which it has retained even now.

Ram Prakash Bambah

Bambah and Saudamini in wedlock.

Soon after the birth of his second daughter in September 1964, Bambah joined Ohio State University (OSU), USA, as professor having got indefinite leave from PU. He spent fruitful years at OSU until his return to PU in 1968. Three of his students received Ph D degrees from OSU in 1965. Bambah also kept close relationship with his colleagues in OSU and made several visits. The Bambahs were known for their warm hospitality on the campus. Their house was open to all University employees for consultation with Saudamini. She was active in the social life of the PU campus and Chandigarh in general, making considerable contributions. She was President of the Chandigarh Chapter of the Indian Society of Blood Transfusion; Member UT Adviser's Advisory Committee; PGI Ethics Committee; Child Welfare Committee and Bhartiya Vidya Bhawan. She was one of the founders of the Ankur School on the PU campus. Saudamini passed away on 14 November 2011 after a brief illness.

Bambah achieved many milestones by virtue of his genius and habit of working hard. He was awarded the Sc D degree by Cambridge University in 1970. In 1971, he was appointed Dean of University Instructions at PU. He was also Vice-Chancellor of PU from 1985 to 1991. He is an Emeritus Professor at PU since 1993. Though Bambah was busy performing administrative duties, his heart was always with mathematics. He guided five Ph D students and had 15 collaborators.

Of course, it is not possible to describe Bambah's numerous invaluable contributions to discussions and seminars, formal or informal, in which he freely suggested problems, gave his guidance and suggestions. Tea time with him has always been replete with discussions of mathematical news and problems. A remarkable fact worth mentioning is that he did not publish jointly with his Ph D students any part of their thesis work, which is normal practice. He has published more than 70 papers in research journals and has edited a book on number theory published by INSA. He has been on the editorial boards of the *Journal of Number Theory*, *Journal of Indian Mathematical Society, Mathematics Student* and *Indian Journal of Pure and Applied Mathematics.*

Bambah has been decorated with many awards and honours. He was President of the Indian Mathematical Society during 1969; Fellow of INSA (1955), Indian Academy of Sciences (1974), National Academy of Sciences (1978) and TWAS (now known as The World Academy of Sciences) (1993). He was the vice-president of INSA during 1979–80; and UGC National fellow for two years (1973–75). He was President of the Mathematical Section of ISCA during 1973 and its General President during 1983–84. He was awarded the Ramanujan Medal of INSA (1979), Distinguished Service Award by the Mathematical Association of India (1984), Meghnad Saha Sward by UGC (1986), Padma Bhushan (1988), Ramanujan Memorial Lecture Award of the Indian Mathematical Society (1993), Ramanujan Birth Centenary Award of ISCA (1994), Jawaharlal Nehru Birth Centenary Lecture Award of ISCA (1997) and Aryabhata Medal of INSA (1998). He has been a member of the Board of Trustees of the

Tribune group of newspapers and chairman of the governing body of Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune. He is the Vice-Chairman of the governing body of the Centre for Research in Rural and Industrial Development (CRRID). He is a member of PU Senate since 2012. On 9 March 2014, PU conferred on him the degree of Doctor of Science (h.c).

An article by Latika Sharma in Souvenir, '*The Making of Modern Punjab – Education, Science & Social Change c.1850–c.2000*', brought out by PU in October 2013, details some events and people who influenced Bambah's life, based on recorded interviews with him.

Research contributions

Bambah attracted some brilliant mathematicians to his department. There was noticeable research activity not only in number theory, but also in other areas like algebra, algebraic number theory, and analysis and applied mathematics. He not only made substantial contributions

Bambah felicitated on his 85th birthday in a symposium organized by IISER Mohali and INSA local chapter.

Bambah on his birthday in 2013 (with the present authors on the extreme left and right).

The Bambahs in Ohio.

Bambah receiving Padma Bhushan.

in number theory, geometry of numbers and discrete geometry, but also established a school which worked and settled some outstanding problems under his guidance/collaboration. We shall briefly describe some of them.

Ramanujan's tau function

Bambah was initiated into research by Chowla, who introduced him to the work of the legendary Indian mathematician Ramanujan (1887–1920). While considering the number of representations of a natural number as the sum of the 24th power of natural numbers, Ramanujan had introduced the function $\tau(n)$ as follows

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\left[(1-x)(1-x^2)...\right]^{24}.$$

So $\tau(n)$ is the coefficient of x^n in the polynomial which results when we take terms up to $1 - x^n$ in the product on the right side of the above equation. Ramanujan worked extensively with this function, made tables of values of $\tau(n)$, found many congruence properties and made some conjectures. Hardy gave a series of lectures on Ramanujan's work during 1936–40. His tenth lecture delivered in 1940 was devoted entirely to tau function. It is not surprising that Bambah started his research on this interesting function and published his first paper in 1946 entitled 'Two congruence properties of Ramanujan's function $\tau(n)$' in the *Journal of* the *London Mathematical Society*.

Twelve more papers by him on this interesting function followed in quick succession. Bambah writes in the biographical memoir of Chowla (published by INSA): 'Because of the later work of

UT Administrator Shivraj V. Patil awards the degree of Doctor of Science to Bambah during the 63rd annual convocation of Panjab University.

many powerful mathematicians, $\tau(n)$ occupies a very central place in mainstream mathematics because of its connection with modular forms, elliptic curves and so on, which play an important role in modern mathematics and in fact, have been crucial in the solution of Fermat's last theorem.'

An intriguing conjecture

Bambah and Chowla (1947) used a straightforward construction to prove the existence of a constant C such that for $x \geq 1$, there is atleast one integer between x and $x + Cx^{1/4}$, which can be expressed as a sum of two squares. They remarked that this is far from the probable truth. But no essential improvement in this result has been obtained so far. Several mathematicians [K. N. Majumdar (1950), S. Uchiyama (1965), P. H. Diananda (1966) and L. J. Mordell (1969)] have worked on the value of C and generalization of this result. Hooley (1971) proved some related results.

Geometry of numbers

The term 'geometry of numbers' was coined by the German mathematician H. Minkowski (1864–1909). Although the problems had been studied by Lagrange, Gauss and other mathematicians and geometrical methods used to solve arithmetical problems, it was Minkowski who observed that geometrical methods can lead to simple proofs of the existence of solutions in integers (not all zeros) of some inequalities. One has only to verify if the n-dimensional body defined by the inequality has certain properties. Soon this developed into a major branch of number theory. Many mathematicians tried their hand on the exciting problems. Bambah was at the right place, amidst the right people, in the 'golden age' of geometry of numbers and made full use of this opportunity.

In an effort to describe his work, we have to introduce some basic terminology. We consider the n-dimensional Euclidean space $\mathbb{R}^n$. A lattice is said to be *admissible* for a set S in $\mathbb{R}^n$, if it has no point other than the origin in the interior of S. The *critical determinant* of S is the infimum of the determinants of lattices admissible for S. A fundamental problem in the geometry of numbers is to determine the critical determinant of a

given set. For convex sets centred at the origin in the plane, general results are known, which at least theoretically give the critical determinant. But for non-convex sets, problems are more difficult. Techniques have to be found and details worked out for different regions. The difficulty increases in higher dimensions both for convex and non-convex sets. For subsets S, T of $\mathbb{R}^n$, (S, T) is called a packing if the sets obtained by translating S through points of T are non-overlapping. This concept is related to the concept of admissibility. Also (S, T) is called a *covering* if each point of $\mathbb{R}^n$ lies in some translate of S through a point of T.

Ph D thesis

In his thesis entitled 'Some results in the geometry of numbers', Bambah developed a method for determining the critical determinant of non-convex star regions with hexagonal symmetry. He used this to obtain the critical determinants of several types of regions and, in particular, of the region $|f(x, y)| \leq 1$, where $f(x, y)$ is a binary cubic form of positive determinant, which was originally determined by Mordell. He also obtained results for special non-homogenous binary quadratic forms and non-homogeneous cubic forms. In 1951, four papers were published from his thesis. One appeared in the *Philosophical Transactions of the Royal Society of London*, two in *Acta Mathematica* and one in *Proceedings of the Cambridge Philosophical Society*.

Coverings

Historically, the theory of packings had attracted more attention than the concept of covering. Bambah's work activated research in this field. When he was research fellow at the National Institute of Sciences, he developed the theory of coverings, answering general questions which are analogues of the corresponding results on critical determinants and packings.

One of the basic problems in the geometry of numbers is the determination of lattice covering density θ_n of the n-dimensional sphere. The result for a circle goes back to Dirichlet (1842). After more than a century, it was Bambah (1954) who obtained the result for three-dimensional spheres and opened

the way for further investigations. Other proofs were given by E. S. Barnes (1956) and L. Few (1956). Bambah also obtained bounds on θ_4 and conjectured its value. This conjecture was proved by B. N. Delone and S. S. Ryshkov 1963 and T. J. Dickson (1967). Ryshkov and E. P. Baranovski (1975) obtained θ_5. Some work has been done in higher dimensions, but the exact values have not been obtained so far.

The first non-trivial lower bounds on θ_n were obtained by Bambah and Davenport in 1952. Bambah and Roth (1952) obtained upper bounds for the lattice covering density of convex bodies which are symmetrical about the coordinate planes.

A natural question is whether the best lattice covering density of a given body equals its best general covering density. Kershner (1939) had proved that this is so for a circle and L. Fejes Tóth gave a proof for two-dimensional symmetric convex domains. Bambah and C. A. Rogers completed this proof in 1952. Simplified proofs were given by Bambah, Rogers and Zassenhaus (1964), and by Bambah and Woods (1968). They also showed that the result holds for a cylinder with symmetric convex domain as base. Their methods led to the study of finite coverings, which has triggered lot of work by other mathematicians.

Bambah, Dumir and Hans-Gill (1977) gave examples to show that there exist star domains (symmetric and asymmetric) in the plane for which the question has a negative answer. Bambah had thought about the idea of construction of the examples while travelling in a bus to Delhi. Earlier, L. Stein (1972) had given examples in R^{10} for symmetric star domains and in R^5 for asymmetric domains.

The question is still open for the three-dimensional spheres. In this direction, Bambah and Woods (1971) showed that double lattices do not give better coverings than the lattices. (The analogous problem for packings is known as Kepler's conjecture and has been settled by T. C. Hales with massive use of high-speed computers.)

Minkowski's conjecture

Minkowski is believed to have conjectured that if L_i, $i = 1, ..., n$, are n real linear forms in n variables having determinant 1 and c_i, $i = 1, ..., n$ are n real numbers, then there exist integral values of the variables satisfying

$$|(L_1 + c_1) \dots (L_n + c_n)| \leq 1/2^n.$$

Minkowski had proved the case $n = 2$ of this conjecture in 1899. It is easy to see that if the conjecture is true, then the result would be best possible. Several mathematicians have worked on this problem. For $n = 2$, many proofs are available. For $n = 3$, the conjecture was proved by Remak (1923) and the proof was simplified by Davenport (1939). This is called the Remak–Davenport method. (Two other proofs were given by Birch and Swinnerton-Dyer (1956), and Narazullaev (1968).) In this method proof can be divided into two parts, which we call parts I and II. Dyson (1948) extended this method, but the proof was difficult. About part I of the proof Dyson remarked, 'The proof borrows weapons from the armory of Topology; purely geometrical methods seem quite inadequate.' Bambah and Woods (1974) gave an elementary proof of part I of Dyson's result, without using strong tools from algebraic topology. (A different proof of this was given by Skubenko in his papers in 1972–76, which also included an initial publication of a proof for $n = 5$.) Bambah and Woods (1980) gave a complete proof of part I for $n = 5$. Since Woods (1965) had already proved part II, Minkowski's conjecture was proved for $n = 5$. C. T. McMullen (2005) proved part I for all n. Since Woods (1972) had already proved part II for $n = 6$, the conjecture followed for $n = 6$. Part II and hence the conjecture has been proved for $n = 7$, 8 by Hans-Gill *et al.* (2009, 2011). Recently, the case $n = 9$ has been settled by Raka and Leetika.

Non-homogeneous quadratic forms: Minkowski's theorem for $n = 2$ can be interpreted as a result of non-homogeneous indefinite binary quadratic forms. In this form its generalization to indefinite quadratic forms in n variables had been formulated and the best possible constants $C_{r,s}$ had been defined. Blaney (1948) proved their existence. Davenport (1948) determined $C_{2,1} = C_{1,2}$

and Birch (1958) obtained $C_{r,r}$ for all r. Watson (1962) determined $C_{r,s}$ for all $n = r + s \geq 21$ and made a conjecture. Under Bambah's guidance and encouragement, his school succeeded in evaluating $C_{r,s}$ for signatures 1, 2, 3, 4, and this was useful for completely settling the problem in collaboration with Woods. Bambah's school also settled the analogous problem of positive values of non-homogeneous indefinite quadratic problems, except for the case of forms of type (1, 4). Best known non-trivial bounds for this case are provided by Madhu Raka and Urmila Rani (1997).

Minkowski's second theorem: This is a generalization of Minkowski's fundamental theorem. Proofs had been given by Minkowski (1896), Davenport (1939), Esterman (1946) and Weyl (1942). But these were not satisfactory. Bambah, Woods and Zassenhaus (1965) gave three proofs which are simpler and more satisfactory. These have been used by Woods (1965) and Mcfeit (1965) to obtain some generalizations. The simplest proof so far has been obtained by Martin Henk in 2002.

Some other contributions: Bambah has also contributed to integer matrices, polar reciprocal convex bodies, divided cells, transference theorems, lower bounds for minimum distance codes, convex bodies with covering property, maximal covering sets, saturated systems and many other topics. In particular, Bambah and Woods (1994) solved a problem of G. Fejes Tóth on the thinnest lattice arrangement of equal spheres in R^3, so that each line meets one of these.

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