RESIDUAL QUOTIENT AND ANNIHILATOR OF INTUITIONISTIC FUZZY SETS OF RING AND MODULE

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ABSTRACT

In this paper, we introduce the concept of residual quotient of intuitionistic fuzzy subsets of ring and module and then define the notion of residual quotient intuitionistic fuzzy submodules , residual quotient intuitionistic fuzzy ideals. We study many properties of residual quotient relating to union, intersection, sum of intuitionistic fuzzy submodules (ideals). Using the concept of residual quotient, we investigate some important characterization of intuitionistic fuzzy annihilator of subsets of ring and module. We also study intuitionistic fuzzy prime submodules with the help of intuitionistic fuzzy annihilators. Many related properties are defined and discussed.

KEYWORDS

Intuitionistic fuzzy (prime) submodule (ideal), residual quotient intuitionistic fuzzy submodules (ideal), intuitionistic fuzzy annihilator, semiprime ring.

1. INTRODUCTION

The concept of intuitionistic fuzzy sets was introduced by Atanassov [1], [2] as a generalization to the notion of fuzzy sets given by Zedah [16]. Biswas was the first to introduce the intuitionistic fuzzification of the algebraic structure and developed the concept of intuitionistic fuzzy subgroup of a group in [5]. Hur and others in [8] defined and studied intuitionistic fuzzy subrings and ideals of a ring. In [7] Davvaz et al. introduced the notion of intuitionistic fuzzy submodules which was further studied by many mathematicians (see [4], [9], [12], [13], [14]).

The correspondence between certain ideals and submodules arising from annihilation plays a vital role in the decomposition theory and Goldie like structures (see [6]). A detailed study of the fuzzification of this and related concepts can be found in [10], [11] and [15]. Intuitionistic fuzzification of such crisp sets leads us to structures that can be termed as intuitionistic fuzzy prime submodules. In this paper, we attempt to define annihilator of an intuitionistic fuzzy subset of a module using the concept of residual quotients and investigate various characteristic of it. This concept will help us to explore and investigate various facts about the intuitionistic fuzzy aspects of associated primes, Godlie like structures and singular ideals.

2. PRELIMINARIES

Throughout this section, R is a commutative ring with unity 1, $1 \neq 0$, M is a unitary R-module and θ is the zero element of M. The class of intuitionistic fuzzy subsets of X is denoted by IFS(X).

Definition (2.1)[4]

Let R be a ring. Then $A \in$ IFS(R) is called an intuitionistic fuzzy ideal of R if for all $x , y \in$ R it satisfies

(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$;

(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$, $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

The class of intuitionistic fuzzy ideals of R is denoted by IFI(R).

Definition (2.2)[4]

An intuitionistic fuzzy set A of an R-module M is called an intuitionistic fuzzy submodule (IFSM) if for all x, $y \in$ M and $r \in$ R , we have

(i) $\mu_A(\theta) = 1$, $\nu_A(\theta) = 0$;

(ii) $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$;

(iii) $\mu_A(rx) \geq \mu_A(x)$, $\nu_A(rx) \leq \nu_A(x)$.

The class of intuitionistic fuzzy submodules of M is denoted by IFM(M).

Definition (2.3)[2, 12]

Let α, β ∈ [0, 1] with α + β ≤ 1. An intuitionistic fuzzy point, written as $x_{(\alpha,\beta)}$, is defined to be an intuitionistic fuzzy subset of X, given by

$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha,\beta) & ; \, if \;\; y = x \\ (0,1) & ; \, if \;\; y \neq x \end{cases}$. We write $x_{(\alpha,\beta)} \in A$ if and only if $x \in C_{(\alpha,\beta)}(A)$,

where $C_{(\alpha,\beta)}(A) = \{x \in X : \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta\}$ is the $(\alpha,\beta)-cut$ set (crisp set) of the intuitionistic fuzzy set A in X.

Definition (2.4)[9]

Let M be an R-module and let A, B ∈ IFM(M). Then the sum A + B of A and B is defined as

$$\mu_{A+B}(x) = \begin{cases} \mu_A(\mathrm{y}) \wedge \mu_B(\mathrm{z}) & ; \, if \;\; x = y+z \\ 0 & ; \, \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{A+B}(x) = \begin{cases} \nu_A(\mathrm{y}) \vee \nu_B(\mathrm{z}) & ; \, if \;\; x = y+z \\ 1 & ; \, \text{otherwise} \end{cases}$$

Then, A + B∈ IFM(M).

Definition (2.5)

Let M be an R-module and let A∈ IFS(R) and B∈ IFM(M). Then the product AB of A and B is defined as

$$\mu_{AB}(x) = \begin{cases} \mu_A(r) \wedge \mu_B(m) & ; \, if \;\; x = rm \\ 0 & ; \, \text{otherwise} \end{cases} \text{ and } \nu_{AB}(x) = \begin{cases} \nu_A(r) \vee \nu_B(m) & ; \, if \;\; x = rm \\ 1 & ; \, \text{otherwise} \end{cases} , r \in R, \; m \in M.$$

Clearly, AB∈ IFM(M).

Definition (2.6) [9, 12]

Let M be an R-module and let A, B ∈ IFM(M). Then the product AB of A and B is defined as

$$\mu_{AB}(x) = \begin{cases} \mu_A(y) \wedge \mu_B(z) & ; \, if \;\; x = yz \\ 0 & ; \, \text{otherwise} \end{cases} \text{ and } \nu_{AB}(x) = \begin{cases} \nu_A(y) \vee \nu_B(z) & ; \, if \;\; x = yz \\ 1 & ; \, \text{otherwise} \end{cases} \text{,where } y, z \in M.$$

Definition (2.7) [3]

An P∈ IFI(R) is called an intuitionistic fuzzy prime ideal of R if for any A, B ∈ IFI(R) the condition AB ⊆ P implies that either A ⊆ P or B ⊆ P.

Definition (2.8)

Let X be a non empty set and A ⊂ X. Then an intuitionistic fuzzy set $\chi_A = \left(\mu_{\chi_A}, \nu_{\chi_A}\right)$ is called an intuitionistic fuzzy characteristic function and is defined as

$$\mu_{\chi_A}(x) = \begin{cases} 1 & ; if\ x \in A \\ 0 & ; if\ x \notin A \end{cases} \quad \text{and} \quad \nu_{\chi_A}(x) = \begin{cases} 0 & ; if\ x \in A \\ 1 & ; if\ x \notin A \end{cases}.$$

Definition (2.9)

Let M be R-module and χ_θ is an IFS on M defined as $\chi_\theta(x) = \left(\mu_{\chi_\theta}(x), \nu_{\chi_\theta}(x)\right)$,

$$where\ \ \mu_{\chi_\theta}(x) = \begin{cases} 1 & ; if\ x = \theta \\ 0 & ; if\ x \neq \theta \end{cases} \quad \text{and} \quad \nu_{\chi_\theta}(x) = \begin{cases} 0 & ; if\ x = \theta \\ 1 & ; if\ x \neq \theta \end{cases}$$

and χ_0 and χ_R are IFSs on R defined by

$\chi_0(r) = \left(\mu_{\chi_0}(r), \nu_{\chi_0}(r)\right)$ and $\chi_R(r) = \left(\mu_{\chi_R}(r), \nu_{\chi_R}(r)\right)$,where

$$\mu_{\chi_0}(r) = \begin{cases} 1 & ; if\ r = 0 \\ 0 & ; if\ r \neq 0 \end{cases} ;\ \nu_{\chi_0}(r) = \begin{cases} 0 & ; if\ r = 0 \\ 1 & ; if\ r \neq 0 \end{cases} \text{ and } \mu_{\chi_R}(r) = 1 ;\ \nu_{\chi_R}(r) = 0,\ \forall\ r \in R.$$

Theorem (2.10)

Let $x \in$ R and α , β ∈ (0,1] with α + β ≤ 1. Then $< x_{(\alpha,\beta)} > = (\alpha,\beta)_{<x>}$,

where $(\alpha,\beta)_{<x>}(y) = \begin{cases} (\alpha,\beta) & ; if\ y \in <x> \\ (0,1) & ; if\ y \notin <x> \end{cases}$ is called the $(\alpha,\beta)-level(or\ cut\ set)$ intuitionistic fuzzy ideal corresponding to $<x>$.

Proof. Case(i) When $y \in <x>$ and let $y = x^n$, for some positive interget n, then

$\mu_{(\alpha,\beta)_{<x>}}(y) = \alpha = \mu_{x_{(\alpha,\beta)}}(x) \leq \mu_{x_{(\alpha,\beta)}}(x^n) = \mu_{x_{(\alpha,\beta)}}(y)$ and

$\nu_{(\alpha,\beta)_{<x>}}(y) = \beta = \nu_{x_{(\alpha,\beta)}}(x) \geq \nu_{x_{(\alpha,\beta)}}(x^n) = \nu_{x_{(\alpha,\beta)}}(y)$

Case(ii) When $y \notin <x>$, then

$\mu_{(\alpha,\beta)_{<x>}}(y) = 0 = \mu_{x_{(\alpha,\beta)}}(y)$ and $\nu_{(\alpha,\beta)_{<x>}}(y) = 1 = \nu_{x_{(\alpha,\beta)}}(y)$.

Thus in both the cases we find that $(\alpha,\beta)_{<x>} \subseteq x_{(\alpha,\beta)}$.

Now $< x_{(\alpha,\beta)} > = \cap\left\{A : A \in IFI(\text{R}) \text{ such that } x_{(\alpha,\beta)} \subseteq A\right\}$ implies that $(\alpha,\beta)_{<x>} = < x_{(\alpha,\beta)} >$.

3. Residual quotient of intuitionistic fuzzy sets of ring and module

Throughout the paper, R is a commutative ring with unity 1, 1 ≠ 0, M is a unitary R-module and θ is the zero element of M. For a subset X of M, recall that < X > denotes the submodules of M generated by X. For any $x \in$ X, we let $< x >$ denote the submodule of M generated by $\{x\}$.

For any submodule N of M we have (N : M) = { r ∣ $r \in$R, rM ⊆ N }.

In this section, we introduce and study the notion of residual quotient of intuitionistic fuzzy subsets of ring and module.

Definition (3.1)

For A, B $\in$ IFS(M) and C$\in$IFS(R), define the residual quotient (A: B) and (A: C) as follows:

$$(A:\mathrm{B}) = \bigcup\{\mathrm{D}:\ \mathrm{D}\in IFS(R) \text{ such that } D\cdot B \subseteq A\} \text{ and}$$
$$(A:\mathrm{C}) = \bigcup\{\mathrm{E}:\ \mathrm{E}\in IFS(M\ ) \text{ such that } C\cdot E \subseteq A\}$$

Clearly, (A: B) $\in$ IFS(R) and (A: C) $\in$ IFS(M).

Theorem (3.2)

For A, B $\in$ IFS(M) and C$\in$IFS(R). Then we have

(i) $(A:B)=\bigcup\{r_{(\alpha,\beta)}:\ r\in R,\alpha,\beta\in[0,1]$ with $\alpha+\beta\le 1$ such that $r_{(\alpha,\beta)}\cdot B\subseteq A\}$ and

(ii) $(A:C)=\bigcup\{x_{(\alpha,\beta)}:\ x\in M,\alpha,\beta\in[0,1]$ with $\alpha+\beta\le 1$ such that $\mathrm{C}\cdot x_{(\alpha,\beta)}\subseteq A\}$.

Proof. (i) We know that

$$\{r_{(\alpha,\beta)}:\ r\in R,\alpha,\beta\in[0,1] \text{ with } \alpha+\beta\le 1 \text{ such that } r_{(\alpha,\beta)}\cdot B\subseteq A\}\in IFS(\mathrm{R})$$

$$\therefore\quad \{r_{(\alpha,\beta)}:\ r\in R,\alpha,\beta\in[0,1] \text{ with } \alpha+\beta\le 1 \text{ such that } r_{(\alpha,\beta)}\cdot B\subseteq A\}$$
$$\subseteq \{\mathrm{D}:\ \mathrm{D}\in IFS(R) \text{ such that } \mathrm{D}\cdot\mathrm{B}\subseteq A\}$$

$$\Rightarrow \bigcup\{r_{(\alpha,\beta)}:\ r\in R,\alpha,\beta\in[0,1] \text{ with } \alpha+\beta\le 1 \text{ such that } r_{(\alpha,\beta)}\cdot B\subseteq A\}$$
$$\subseteq \bigcup\{\mathrm{D}:\ \mathrm{D}\in IFS(R) \text{ such that } \mathrm{D}\cdot\mathrm{B}\subseteq A\} = (A:B).$$

Let $\mathrm{D}\in \mathrm{IFS(R)}$ such that $\mathrm{D}\cdot\mathrm{B}\subseteq \mathrm{A}$.

Let $r\in \mathrm{R}$ and $\mathrm{D}(r) = (\alpha,\beta)$, i.e., $\mu_D(r)=\alpha$ and $\nu_D(r)=\beta$.

Now, $\left(r_{(\alpha,\beta)}B\right)(x)=\left(\mu_{r_{(\alpha,\beta)}B}(x),\ \nu_{r_{(\alpha,\beta)}B}(x)\right)$, where

$$\begin{aligned}
\mu_{r_{(\alpha,\beta)}B}(x) &= \vee\{\mu_{r_{(\alpha,\beta)}}(s)\wedge\mu_B(y):\ s\in R,\ y\in M,\ sy=x\}\\
&\le \vee\{\mu_D(r)\wedge\mu_B(y):\ y\in M,\ ry=x\}\ [\ \because \mu_{r_{(\alpha,\beta)}}(s)\le\mu_D(r)=\alpha\ ]\\
&= \vee\{\mu_D(s)\wedge\mu_B(y):\ s\in R,\ y\in M,\ sy=x\}\\
&= \mu_{DB}(x)\\
&\le \mu_A(x)
\end{aligned}$$

i.e., $\mu_{r_{(\alpha,\beta)}B}(x)\le\mu_A(x),\ \forall\ x\in M$.

Similarly, we can show that $\nu_{r_{(\alpha,\beta)}B}(x)\ge\nu_A(x),\ \forall\ x\in M$. Thus, $r_{(\alpha,\beta)}B\subseteq A$.

So, (A:B) $\subseteq\bigcup\{r_{(\alpha,\beta)}:\ r\in R,\alpha,\beta\in[0,1]$ with $\alpha+\beta\le 1$ such that $r_{(\alpha,\beta)}B\subseteq A\}$

Hence (A:B) $=\bigcup\{r_{(\alpha,\beta)}:\ r\in R,\alpha,\beta\in[0,1]$ with $\alpha+\beta\le 1$ such that $r_{(\alpha,\beta)}B\subseteq A\}$.

(ii) The proof is similar to (i).

Theorem (3.3) For A, B $\in$ IFS(M) and C$\in$IFS(R). Then we have

(i) $(\mathrm{A}:\mathrm{B})\cdot\mathrm{B}\subseteq\mathrm{A}$;

(ii) $\mathrm{C}\cdot(\mathrm{A}:\mathrm{C})\subseteq\mathrm{A}$;

(iii) $\mathrm{C}\cdot\mathrm{B}\subseteq\mathrm{A} \iff \mathrm{C}\subseteq(\mathrm{A}:\mathrm{B}) \iff \mathrm{B}\subseteq(\mathrm{A}:\mathrm{C})$.

Proof. (i) Now, $\big((\mathrm{A}:\mathrm{B})\cdot\mathrm{B}\big)(x)=\big(\mu_{(\mathrm{A:B})\cdot\mathrm{B}}(x),\ \nu_{(\mathrm{A:B})\cdot\mathrm{B}}(x)\big)$, where

$$\begin{aligned}
\mu_{(\mathrm{A:B})\cdot\mathrm{B}}(x) &= \vee\left\{\mu_{(\mathrm{A:B})}(r)\wedge\mu_B(y)\mid\ r\in R,\ y\in M,\ ry=x\right\}\\
&= \vee\left\{\vee\left\{\mu_D(r)\mid\ \mathrm{D}\in IFS(\mathrm{R}),\mathrm{D}\cdot\mathrm{B}\subseteq A\right\}\wedge\mu_B(y)\mid r\in R,\ y\in M,\ ry=x\right\}\\
&= \vee\left\{\mu_D(r)\wedge\mu_B(y)\mid\ \mathrm{D}\in IFS(\mathrm{R}),\ \mathrm{D}\cdot\mathrm{B}\subseteq A,\ \ y\in M,\ ry=x\right\}\\
&\le \vee\left\{\mu_{D\cdot B}(ry)\mid\ \mathrm{D}\in IFS(\mathrm{R}),\ \mathrm{D}\cdot\mathrm{B}\subseteq A,\ \ y\in M,\ ry=x\right\}\\
&\le \vee\left\{\mu_A(ry)\mid\ \ y\in M,\ ry=x\right\}\\
&= \mu_A(x).
\end{aligned}$$

Thus, $\mu_{(\mathrm{A:B})\cdot\mathrm{B}}(x)\le\mu_A(x),\ \forall\ x\in M$.

Similarly, we can show that $\nu_{(\mathrm{A:B})\cdot\mathrm{B}}(x)\ge\nu_A(x),\ \forall\ x\in M$.

Hence $(\mathrm{A}:\mathrm{B})\cdot\mathrm{B}\subseteq\mathrm{A}$.

(ii) The proof is similar to (i)

(iii) This is an immediate consequences of (i) and (ii).

Theorem (3.4)

For $\mathrm{A_i}$ (i $\in$ J) , B$\in$IFS(M) and C$\in$IFS(R). Then we have

(i) $\left(\bigcap_{i\in J}A_i\right):\mathrm{B}=\bigcap_{i\in J}(A_i:B)$ (ii) $\left(\bigcap_{i\in J}A_i\right):\mathrm{C}=\bigcap_{i\in J}(A_i:C)$.

Proof .(i)
$$\begin{aligned}
\left(\bigcap_{i\in J}A_i\right):\mathrm{B} &= \bigcup\left\{\mathrm{D}:\ \mathrm{D}\in IFS(R)\text{ such that }D\cdot B\subseteq\bigcap_{i\in J}A_i\right\}\\
&= \bigcup\left\{\mathrm{D}:\ \mathrm{D}\in IFS(R)\text{ such that }D\cdot B\subseteq A_i\ ,\ \forall i\in J\right\}\\
&= \bigcup\left\{\mathrm{D}:\ \mathrm{D}\in IFS(R)\text{ such that }D\subseteq (A_i: B)\ ,\ \forall i\in J\right\}\\
&= \bigcup\left\{\mathrm{D}:\ \mathrm{D}\in IFS(R)\text{ such that }D\subseteq\bigcap_{i\in J}(A_i: B)\right\}\\
&\subseteq \bigcap_{i\in J}(A_i: B).
\end{aligned}$$

Thus $\left(\bigcap_{i\in J}A_i\right):\mathrm{B}\subseteq\bigcap_{i\in J}(A_i: B)$.

By previous theorem, we have

$\left(\bigcap_{i\in J}(\mathrm{A}_i:\mathrm{B})\cdot\mathrm{B}\right)(x)=\left(\mu_{\bigcap_{i\in J}(\mathrm{A}_i:\mathrm{B})\cdot\mathrm{B}}(x),\ \nu_{\bigcap_{i\in J}(\mathrm{A}_i:\mathrm{B})\cdot\mathrm{B}}(x)\right)$, where

$$\mu_{\bigcap_{i\in J}(A_i:B)\cdot B}(x)=\vee\left\{\left(\bigcap_{i\in J}\mu_{(A_i:B)}(r)\right)\wedge\mu_B(y)\mid r\in R,\ y\in M,\ ry=x\right\}$$
$$\leq\vee\left\{\underset{i\in J}{\wedge}\left(\mu_{(A_i:B)}(r)\right)\wedge\mu_B(y)\mid r\in R,\ y\in M,\ ry=x\right\}$$
$$\leq\vee\left\{\underset{i\in J}{\wedge}\left(\mu_{(A_i:B)}(ry)\right)\mid r\in R,\ y\in M,\ ry=x\right\}$$
$$\leq\vee\left\{\underset{i\in J}{\wedge}\mu_{A_i}(ry)\mid r\in R,\ y\in M,\ ry=x\right\}$$
$$=\mu_{\bigcap_{i\in J}A_i}(x)$$

i.e., $\mu_{\bigcap_{i\in J}(A_i:B)\cdot B}(x)\leq\mu_{\bigcap_{i\in J}A_i}(x),\ \forall\ x\in M.$

Similarly, we can show that $\nu_{\bigcap_{i\in J}(A_i:B)\cdot B}(x)\geq\nu_{\bigcap_{i\in J}A_i}(x),\ \forall\ x\in M.$

Thus, $\bigcap_{i\in J}(A_i:B)\cdot B\subseteq\bigcap_{i\in J}A_i$ and so $\bigcap_{i\in J}(A_i:B)\subseteq\left(\bigcap_{i\in J}A_i\right):B$ ....(2)

From (1) and (2) we get $\bigcap_{i\in J}(A_i:B)=\left(\bigcap_{i\in J}A_i\right):B.$

(ii) The proof is similar to (i).

Theorem (3.5)

For A, B $\in$ IFS(M) and C$\in$ IFS(R)

(i) If $A\in IFM(M)$, then $(A:B)=\bigcup\{D:\ D\in IFI(R)$ such that $D\cdot B\subseteq A\}$

(ii) If $C\in IFI(R)$, then $(A:C)=\bigcup\{E:\ E\in IFM(M)$ such that $C\cdot E\subseteq A\}$

Proof. (i) Clearly, $\bigcup\{D:\ D\in IFI(R)\text{ s.t }D\cdot B\subseteq A\}\subseteq\bigcup\{D:\ D\in IFS(R)\text{ s.t }D\cdot B\subseteq A\}=(A:B)$.

Let $r\in R,\alpha,\beta\in(0,1]$ with $\alpha+\beta\leq1$ such that $r_{(\alpha,\beta)}\cdot B\subseteq A.$

Let $D=\langle r_{(\alpha,\beta)}\rangle$. Then $\langle r_{(\alpha,\beta)}\rangle\cdot B=(\alpha,\beta)_{<r>}\cdot B.$

Again, $\mu_{(\alpha,\beta)_{<r>}B}(x)=\vee\left\{\mu_{(\alpha,\beta)_{<r>}}(s)\wedge\mu_B(y)\mid r\in R,\ y\in M,\ sy=x\right\}$
$$=\vee\{\alpha\wedge\mu_A(y)\mid s\in\langle r\rangle,\ y\in M,\ sy=x\}$$
$$\leq\vee\left\{\mu_{r_{(\alpha,\beta)}B}(ry)\mid t\in R,\ y\in M,\ \mathrm{t}(ry)=x\right\}$$
$$\leq\vee\{\mu_A(ry)\mid t\in R,\ y\in M,\ \mathrm{t}(ry)=x\}$$
$$\leq\vee\{\mu_A(\mathrm{t}(ry))\mid t\in R,\ y\in M,\ \mathrm{t}(ry)=x\}$$
$$=\mu_A(x).$$

Thus, $\mu_{(\alpha,\beta)_{<r>}B}(x)\leq\mu_A(x)$.Similarly, we can show that $\nu_{(\alpha,\beta)_{<r>}B}(x)\geq\nu_A(x),\forall x\in M.$

Therefore, we have $(\alpha,\beta)_{<r>}\cdot B\subseteq A.$

Hence $\bigcup\{D:\ D\in IFI(R)$ such that $D\cdot B\subseteq A\}\supseteq(A:B)$

Hence $(A:B)=\bigcup\{D:\ D\in IFI(R)$ such that $D\cdot B\subseteq A\}.$

(ii) The proof is similar to (i).

Theorem (3.6)

For A∈IFM(M), B∈IFS(M) and C∈IFI(R). Then

(i) (A:B)∈IFI(R) and (ii) (A:C)∈IFM(M).

Proof. (i) Since $\chi_0 A \subseteq \chi_\theta \subseteq A$, so $\chi_0 \subseteq (A: B)$. Let $r_1, r_2, r, s \in R$ be any elements. Then,

$$\mu_{(A:B)}(r_1) \wedge \mu_{(A:B)}(r_2)$$
$$= \left(\vee\left\{\mu_{A_1}(r_1) : A_1 \in IFI(R), A_1 A \subseteq B\right\}\right) \wedge \left(\vee\left\{\mu_{A_2}(r_2) : A_2 \in IFI(R), A_2 A \subseteq B\right\}\right)$$
$$= \vee\left\{\mu_{A_1}(r_1) \wedge \mu_{A_2}(r_2) : A_1, A_2 \in IFI(R), A_1 A \subseteq B, A_2 A \subseteq B\right\}$$
$$\leq \vee\left\{\mu_{(A_1+A_2)}(r_1) \wedge \mu_{(A_1+A_2)}(r_2) : A_1, A_2 \in IFI(R), A_1 A \subseteq B, A_2 A \subseteq B\right\}$$
$$\leq \vee\left\{\mu_{(A_1+A_2)}(r_1 - r_2) : A_1 + A_2 \in IFI(R), A_1 A + A_2 A \subseteq (A_1 + A_2)A \subseteq B + B = B\right\}$$
$$\leq \vee\left\{\mu_D(r_1 - r_2) : D \in IFI(R), DA \subseteq B\right\}$$
$$= \mu_{(A:B)}(r_1 - r_2).$$

Thus, $\mu_{(A:B)}(r_1 - r_2) \geq \mu_{(A:B)}(r_1) \wedge \mu_{(A:B)}(r_2)$.

Similarly, we can show that $\nu_{(A:B)}(r_1 - r_2) \leq \nu_{(A:B)}(r_1) \wedge \nu_{(A:B)}(r_2)$.

Again, $\mu_{(A:B)}(sr) = \vee\{\mu_D(sr) : D \in IFI(R), DA \subseteq B\} \geq \vee\{\mu_D(r) : D \in IFI(R), DA \subseteq B\} = \mu_{(A:B)}(r)$.

Thus, $\mu_{(A:B)}(sr) \geq \mu_{(A:B)}(r)$. *Similarly*, we can show that $\nu_{(A:B)}(sr) \leq \nu_{(A:B)}(r)$, $\forall\, r, s \in R$.

Hence $(A:B) \in IFI(R)$.

(ii) The proof is similar to (i).

Let A, B∈IFM(M) and C∈IFI(R). Then (A:C) is called the residual quotient intuitionistic fuzzy submodule of A and C and (A:B) is called the residual quotient intuitionistic fuzzy ideal of A and B respectively.

Theorem (3.7)

For A, B_i ∈IFS(M) and C_i∈IFS(R) for i ∈ J. Then we have

(i) $A : \left(\bigcup_{i \in J} B_i\right) = \bigcap_{i \in J} (A : B_i)$ (ii) $A : \left(\bigcup_{i \in J} C_i\right) = \bigcap_{i \in J} (A : C_i)$

Proof .(i) By definition (3.1)

$$A : \left(\bigcup_{i \in J} B_i\right) = \bigcup\left\{\mathrm{D} :\ \mathrm{D} \in IFS(R) \text{ such that } D \cdot \left(\bigcup_{i \in J} B_i\right) \subseteq A\right\}$$
$$= \bigcup\left\{\mathrm{D} :\ \mathrm{D} \in IFS(R) \text{ such that } \bigcup_{i \in J} (D : B_i) \subseteq A\right\}$$
$$\subseteq \bigcup\{\mathrm{D} :\ \mathrm{D} \in IFS(R) \text{ such that } (D : B_i) \subseteq A\},\ \forall i \in J$$
$$= (A : B_i),\ \forall i \in J.$$

Therefore, $A : \left(\bigcup_{i \in J} B_i\right) \subseteq \bigcap_{i \in J} (A : B_i)$.

By Theorem (3.3)

$$\left(\bigcap_{i\in J}(A:B_i)\right)\cdot\left(\bigcup_{i\in J}B_i\right)=\bigcup_{j\in J}\left(\left(\bigcap_{i\in J}(A:B_i)\right)\cdot B_j\right)\subseteq\bigcup_{j\in J}\left((A:B_j)\cdot B_j\right)\subseteq\bigcup_{j\in J}A=A.$$

Thus $\bigcap_{i\in J}(A:B_i)\subseteq A:\left(\bigcup_{i\in J}B_i\right)$. Hence $A:\left(\bigcup_{i\in J}B_i\right)=\bigcap_{i\in J}(A:B_i)$.

Lemma (3.8)

Let M be a R-module, then $(\chi_\theta:\chi_\theta)=\chi_R$.

Proof . Since $(\chi_\theta:\chi_\theta)=\bigcup\{\mathrm{D}:\ \mathrm{D}\in IFS(R)\ \text{ such that }\ D\cdot\chi_\theta\subseteq\chi_\theta\}\subseteq\chi_R$.

We claim that $\chi_R\subseteq(\chi_\theta:\chi_\theta)$. For this, we first show that $\chi_R\chi_\theta=\chi_\theta$.

Now, $\mu_{\chi_R\chi_\theta}(x)=\vee\{\chi_R(r)\wedge\chi_\theta(m):\ r\in R,\ m\in M,\ rm=x\}$ and

$\nu_{\chi_R\chi_\theta}(x)=\wedge\{\chi_R(r)\vee\chi_\theta(m):\ r\in R,\ m\in M,\ rm=x\}$.

Now, $\mu_{\chi_R\chi_\theta}(x)=\vee\{\chi_R(r)\wedge\chi_\theta(m):\ r\in R,\ m\in M,\ rm=x\}$

$$=\vee\{\chi_\theta(m):\ r\in R,\ m\in M,\ rm=x\}$$

$$=\begin{cases}1 & ;\ if\ x=\theta \qquad [\because\ if\quad x=\theta\Rightarrow one\ m=\theta]\\ 0 & ;\ if\ x\neq\theta \qquad [\because\ if\quad x\neq\theta\ \Rightarrow\ m\ \neq\theta]\end{cases}$$

Also, $\nu_{\chi_R\chi_\theta}(x)=\wedge\{\chi_R(r)\vee\chi_\theta(m):\ r\in R,\ m\in M,\ rm=x\}$

$$=\wedge\{\chi_\theta(m):\ r\in R,\ m\in M,\ rm=x\}$$

$$=\begin{cases}0 & ;\ if\ x=\theta\\ 1 & ;\ if\ x\neq\theta\end{cases}.$$

Thus, $\chi_R\chi_\theta(x)=\chi_\theta(x)$.

So, $\chi_R\subseteq\bigcup\{\mathrm{D}:\ \mathrm{D}\in IFS(R),\ \ \mathrm{D}\chi_\theta\subseteq\chi_\theta\}=(\chi_\theta:\chi_\theta)$.

Hence $(\chi_\theta:\chi_\theta)=\chi_R$.

Lemma (3.9)

Let M be a R-module, then $\chi_0\subseteq(\chi_\theta:\mathrm{A}),\ \forall\ \mathrm{A}\in IFS(M)$.

Proof. We show that $\chi_0A\subseteq\chi_\theta\ \ \forall\ \ \mathrm{A}\in IFS(M)$.

Now, $\mu_{\chi_0A}(x)=\vee\{\mu_{\chi_0}(r)\wedge\mu_A(m):\ r\in R,\ m\in M,\ rm=x\}$

When $x\neq\theta\ \Rightarrow\ \ r\neq 0\ ;\ \forall\ r\in R$, such that $rm=x$

$\Rightarrow\ \mu_{\chi_0}(r)=0\ \forall\ r\in R$, such that $rm=x$. So, $\mu_{\chi_0A}(x)=0=\mu_{\chi_\theta}(x)$.

When $x=\theta\ \Rightarrow\mu_{\chi_0A}(\theta)\leq 1=\mu_{\chi_\theta}(\theta)$. Thus, $\mu_{\chi_0A}(x)\leq\mu_{\chi_\theta}(x)$.

Similarly, we can show that $\nu_{\chi_0A}(x)\geq\nu_{\chi_\theta}(x)$. Therefore, $\chi_0A\subseteq\chi_\theta$.

Hence $\chi_0\subseteq\bigcup\{\mathrm{D}\ :\ \mathrm{D}\in IFS(R)$ such that $\mathrm{D.A}\subseteq\chi_\theta\}=(\chi_\theta:A)$.

Lemma (3.10)

Let M be a R-module and A, B$\in$IFS(M). If $A \subseteq B$, then $(\chi_\theta : B) \subseteq (\chi_\theta : A)$.

Proof. Let A, B $\in IFS(\mathrm{M})$, $D \in IFS(\mathrm{R})$. Then $DA(x) = (\mu_{DA}(x),\ \nu_{DA}(x))$, where

$\mu_{DA}(x) = \vee\{\mu_D(r) \wedge \mu_A(m):\ r \in R,\ m \in M,\ rm = x\}$ and

$\nu_{DA}(x) = \wedge\{\nu_D(r) \vee \nu_A(m):\ r \in R,\ m \in M,\ rm = x\}$.

Now, $\mu_D(r) \wedge \mu_A(m) \leq \mu_D(r) \wedge \mu_B(m)$.

Therefore, $\mu_{DA}(x) = \vee\{\mu_D(r) \wedge \mu_A(m):\ r \in R,\ m \in M,\ rm = x\}$

$\leq \vee\{\mu_D(r) \wedge \mu_B(m):\ r \in R,\ m \in M,\ rm = x\}$

$= \mu_{DB}(x)$.

Similarly, we can show that $\nu_{DA}(x) \geq \nu_{DB}(x)$. Thus $DA \subseteq DB$.

So, $DB \subseteq \chi_\theta \ \Rightarrow\ DA \subseteq \chi_\theta$.

$\therefore\ \bigcup\{D : D \in IFS(R)$ such that $DB \subseteq \chi_\theta\} \subseteq \bigcup\{D : D \in IFS(R)$ such that $DA \subseteq \chi_\theta\}$

$\Rightarrow\ (\chi_\theta : B) \subseteq (\chi_\theta : A)$.

Theorem (3.11)

Let M is a R-module and A, B $\in$ IFM(M), then $(\chi_\theta : A + B) = (\chi_\theta : A) \cap (\chi_\theta : B)$.

Proof. Since A, B $\in$ IFM(M) $\Rightarrow$ A + B $\in$ IFM(M), we have

$\mu_{A+B}(x) = \underset{x=y+z}{\vee}\{\mu_A(y) \wedge \mu_B(z)\} \geq \mu_A(x) \wedge \mu_B(\theta) = \mu_A(x)$ and

$\nu_{A+B}(x) = \underset{x=y+z}{\wedge}\{\nu_A(y) \vee \nu_B(z)\} \leq \nu_A(x) \vee \nu_B(\theta) = \nu_A(x),\ \forall\ x \in M$.

This implies that $A \subseteq A + B$ and $B \subseteq A + B$.

$So, (\chi_\theta : A + B) \subseteq (\chi_\theta : A)$ and $(\chi_\theta : A + B) \subseteq (\chi_\theta : B)$

$\Rightarrow\ (\chi_\theta : A + B) \subseteq (\chi_\theta : A) \cap (\chi_\theta : B)$.

Now, $(\chi_\theta : A) \cap (\chi_\theta : B)$

$= (\cup\{A_1 \mid A_1 \in IFI(R), A_1A \subseteq \chi_\theta\}) \cap (\cup\{B_1 \mid B_1 \in IFI(R), B_1B \subseteq \chi_\theta\})$

$= \cup\{A_1 \cap B_1 \mid A_1, B_1 \in IFI(R), A_1A \subseteq \chi_\theta, B_1B \subseteq \chi_\theta\}$

$\subseteq \cup\{C \mid C = A_1 \cap B_1 \in IFI(R), CA \subseteq \chi_\theta, \mathrm{C}B \subseteq \chi_\theta\}$

$\subseteq \cup\{C \mid C = A_1 \cap B_1 \in IFI(R), C(A + B) \subseteq CA + CB \subseteq \chi_\theta\}$

$= \cup\{C \mid C \in IFI(R), C(A + B) \subseteq \chi_\theta\}$

$= (\chi_\theta : A + B)$.

Therefore, $(\chi_\theta : A) \cap (\chi_\theta : B) \subseteq (\chi_\theta : A + B)$.

Hence $(\chi_\theta : A + B) = (\chi_\theta : A) \cap (\chi_\theta : B)$.

Theorem (3.12)

Let M is a R-module and $A_i \in$ IFM(M), $i \in J$. Then $\left(\chi_\theta : \sum_{i\in J} A_i\right) = \bigcap_{i\in J}(\chi_\theta : A_i)$.

Proof. The proof is similar to above theorem (3.11).

4. Annihilator of intuitionistic fuzzy subset of a ring and module

In this section we study annihilator of intuitionistic fuzzy subset of rings and modules in term of residual quotient and investigate various characteristic of it.

Definition (4.1) Let M be a R-module and A∈ IFS(M), then the annihilator of A is denoted by ann(A) and is defined as: $ann(A) = \bigcup \{B : B \in IFS(R) \text{ such that } BA \subseteq \chi_\theta\}$.

Note that $(\chi_\theta : A) = ann(A)$.

Lemma (4.2) Let M be a R-module, then $ann(\chi_\theta) = \chi_R$.

Proof. The result follows from lemma (3.8) as $ann(\chi_\theta) = (\chi_\theta : \chi_\theta) = \chi_R$.

Lemma (4.3) Let M be a R-module and A∈ IFS(M), then $\chi_0 \subseteq ann(A)$.

Proof. The result follows from lemma (3.9) as $\chi_0 \subseteq (\chi_\theta : A) = ann(A)$.

Lemma (4.4) Let M be a R-module and A, B∈ IFS(M). If $A \subseteq B$, then $ann(B) \subseteq ann(A)$.

Proof. The result follows from lemma (3.10). If $A \subseteq B$ then

$ann(B) = (\chi_\theta : B) \subseteq (\chi_\theta : A) = ann(A)$.

Theorem (4.5) Let M be a R-module and A∈ IFS(M). Then

$ann(A) = \bigcup \{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in [0,1] \text{ with } \alpha + \beta \le 1 \text{ such that } r_{(\alpha,\beta)} A \subseteq \chi_\theta\}$

Proof. The result follows from Theorem (3.2)

Theorem (4.6) Let M be a R-module and A∈ IFS(M). Then $ann(A)A \subseteq \chi_\theta$.

Proof. Follows from Theorem (3.3) (i) by taking $A = \chi_\theta$ and $B = A$.

Corollary (4.7) If A∈ IFS(M) be such that $\mu_A(\theta) = 1$ and $\nu_A(\theta) = 0$, then $ann(A)A = \chi_\theta$.

Proof. By Lemma (4.3) we have $\chi_0 \subseteq ann(A)$

$\Rightarrow \quad \mu_{\chi_0}(0) \le \mu_{ann(A)}(0)$ and $\nu_{\chi_0}(0) \ge \nu_{ann(A)}(0)$

i.e., $1 \le \mu_{ann(A)}(0)$ and $0 \ge \nu_{ann(A)}(0)$

$\Rightarrow \quad \mu_{ann(A)}(0) = 1$ and $\nu_{ann(A)}(0) = 0$.

Now, $\mu_{ann(A)A}(\theta) = \vee\{\mu_{ann(A)}(r) \wedge \mu_A(m) : r \in R,\ m \in M,\ rm = \theta\}$

$\geq \mu_{ann(A)}(0) \wedge \mu_A(\theta)$

$= 1 \wedge 1 = 1$

i.e., $\mu_{ann(A)A}(\theta) = 1$. Similarly, we can show that $\nu_{ann(A)A}(\theta) = 0$.

Therefore, $\chi_\theta \subseteq ann(A)A$. Hence by Theorem (4.6) we get

$ann(A)A = \chi_\theta$.

Note (4.8) If $A \in IFM(M)$, then $ann(A)A = \chi_\theta$.

Theorem (4.9) Let M is a R-module and $B \in IFS(R)$, $A \in IFS(M)$ such that $BA \subseteq \chi_\theta$ if and only if $B \subseteq ann(A)$.

Proof. By definition of annihilator $BA \subseteq \chi_\theta \Rightarrow B \subseteq ann(A)$.
Conversely, let $B \subseteq ann(A) \Rightarrow BA \subseteq ann(A)A \subseteq \chi_\theta$.

Corollary (4.10) If in the above theorem (3.10) $\mu_B(0) = 1$, $\nu_B(0) = 0$ and $\mu_A(\theta) = 1$, $\nu_A(\theta) = 0$, then $BA = \chi_\theta$ if and only if $B \subseteq ann(A)$.

Theorem (4.11) Let M is a R-module and $A, B \in IFS(M)$. Then the following conditions are equivalent:

(*i*) $ann(B) = ann(A)$, for all $B \subseteq A$, $B \neq \chi_\theta$.

(*ii*) $CB \subseteq \chi_\theta$ implies $CA \subseteq \chi_\theta$, for all $B \subseteq A$, $B \neq \chi_\theta$, $C \in IFS(R)$.

Proof. For (i) $\Rightarrow$ (ii) Let $CB \subseteq \chi_\theta$. Then by theorem (3.11) we have
$C \subseteq ann(B) = ann(A)$ (by (i)). Again by the same theorem we have $CA \subseteq \chi_\theta$.
For (ii) $\Rightarrow$ (i) By theorem (4.6) we have $ann(B)B \subseteq \chi_\theta$.
So (ii) implies $ann(B)A \subseteq \chi_\theta$ where $B \subseteq A$, $B \neq \chi_\theta$.
By theorem (4.8) $ann(B) \subseteq ann(A)$.
Also, $B \subseteq A \Rightarrow ann(A) \subseteq ann(B)$. Thus $ann(A) = ann(B)$.

Corollary (4.12) If in the above theorem $\mu_A(\theta) = 1$, $\nu_A(\theta) = 0$ and $\mu_B(\theta) = 1$, $\nu_B(\theta) = 0$. Then the above theorem can be stated as: The following conditions are equivalents:

(*i*) $ann(B) = ann(A)$, for all $B \subseteq A$, $B \neq \chi_\theta$.

(*ii*) $CB = \chi_\theta$ implies $CA = \chi_\theta$, for all $B \subseteq A$, $B \neq \chi_\theta$, $C \in IFS(R)$ with $\mu_C(0) = 1$, $\nu_C(0) = 0$.

Theorem (4.13) Let M is a R-module and $A \in IFS(M)$. Then
$ann(A) = \bigcup\{B : B \in IFI(R) \text{ such that } BA \subseteq \chi_\theta\}$, where IFI(R) is the set of intuitionistic fuzzy ideals of R.

Proof. It follows from Theorem (3.5)(i)

Theorem (4.14) Let M is a R-module and $A \in IFS(M)$. Then $ann(A) \in IFI(R)$.
Proof. It follows from Theorem (3.6)

Theorem (4.15) Let M is a R-module and $A_i \in$ IFS(M), $i \in \Lambda$. Then

$$ann\left(\bigcup_{i\in\Lambda} A_i\right) = \bigcap_{i\in\Lambda} ann(A_i).$$

Proof. It follows from Theorem (3.7)(i)

Theorem (4.16) Let M is a R-module and $A_i \in$ IFM(M) for $i \in J$, then

$$ann\left(\sum_{i\in J} A_i\right) = \bigcap_{i\in J} ann(A_i).$$

Proof. It follows from Theorem (3.12)

Definition (4.17) Let M be R-module. Then $A\in$ IFS(M) is said to be faithful if ann(A) = χ_0.

Lemma (4.18) Let $A\in$ IFS(M) be faithful, where M is R-module. If R is non-zero then A $\neq \chi_\theta$.

Proof. Since A is faithful $\Rightarrow$ ann(A) = χ_0.

If A = χ_θ then ann(A) = ann(χ_θ) = χ_R. Thus we have $\chi_0 = \chi_R \Rightarrow$ R = {0}, a contradiction. Therefore, A $\neq \chi_\theta$.

Theorem (4.19) Let $A \in$ IFS(R) with $\mu_A(0) = 1$, $\nu_A(0) = 0$. Then $A \subseteq$ ann(ann(A)) and ann (ann(ann(A))) = ann(A).

Proof. Let A be an intuitionistic fuzzy subset of R-module R. Then by corollary (4.7), we have ann(A)A = χ_0.

By theorem (4.9), we have $A \subseteq$ ann(ann(A)) ....(1)
$\Rightarrow$ ann(ann(ann(A))) $\subseteq$ ann(A) [using lemma (4.4)]
Again using (1) : ann(A) $\subseteq$ ann(ann(ann(A))).
So, ann (ann(ann(A))) = ann(A).

Theorem(3.20) Let $A \in$ IFS(M). Then

$C_{(\alpha,\beta)}(ann(A)) \subseteq ann(C_{(\alpha,\beta)}(A)), \forall\ \alpha, \beta \in (0,1]$ with $\alpha + \beta \leq 1$.

Proof. Let $x \in C_{(\alpha,\beta)}(ann(A))$. *Then* $\mu_{ann(A)}(x) \geq \alpha > 0$ and $\nu_{ann(A)}(x) \leq \beta < 1$

$\Rightarrow \vee\{\mu_B(x) : B \in IFI(R), BA \subseteq \chi_\theta\} \geq \alpha$ and $\wedge\{\nu_B(x) : B \in IFI(R), BA \subseteq \chi_\theta\} \leq \beta$

$\Rightarrow \mu_B(x) \geq \alpha$ and $\nu_B(x) \leq \beta$ for some $B \in IFI(R)$ with $BA \subseteq \chi_\theta$.

If $x \notin ann(C_{(\alpha,\beta)}(A))$ then $\exists$'s some $y \in C_{(\alpha,\beta)}(A)$ such that $xy \neq \theta$.

Now, $\mu_{BA}(xy) \geq \mu_B(x) \wedge \mu_A(y) \geq \alpha > 0$ and $\nu_{BA}(xy) \leq \nu_B(x) \vee \nu_A(y) \leq \beta < 1$,

which is a contradiction. Hence $C_{(\alpha,\beta)}(ann(A)) \subseteq ann(C_{(\alpha,\beta)}(A))$.

Definition (4.21) $A\in$ IFI(R) is said to be an intuitionistic fuzzy dense ideal if ann(A) = χ_0 .

Definition (4.22) $A\in$ IFI(R) is called intuitionistic fuzzy semiprime ideal of R if for any IFI B of R such that $B^2 \subseteq A$ implies that $B \subseteq A$.

Theorem (4.23) If A is an IFI of a semi prime ring R, then $A \cap ann(A) = \chi_0$ and $A + ann(A)$ is an intuitionistic fuzzy dense ideal of R.

Proof. Since $A \cap ann(A) \subseteq A$, $A \cap ann(A) \subseteq ann(A)$ so $(A \cap ann(A))^2 \subseteq A\, ann(A) \subseteq \chi_0$.
Now R is a semiprime ring and it implies that 0 is a semiprime ideal of R so χ_0 is an intuitionistic fuzzy semi prime ideal of R.
Also, $(A \cap ann(A))^2 \subseteq \chi_0 \Rightarrow A \cap ann(A) \subseteq \chi_0$ and hence $A \cap ann(A) = \chi_0$.
Hence $ann(A + ann(A)) = ann(A) \cap ann(ann(A)) = \chi_0$ proving thereby $A + ann(A)$ is an intuitionistic fuzzy dense ideal of R.

Theorem (4.24) Let A be a non-zero intuitionistic fuzzy ideal of a prime ring R with $\mu_A(0) = 1$, $\nu_A(0) = 0$. Then A is an intuitionistic fuzzy dense ideal of R.
Proof. Now, $A\, ann(A) = \chi_0 \Rightarrow ann(A) = \chi_0$ or $A = \chi_0$. But $A \neq \chi_0$ so $ann(A) = \chi_0$.
Hence A is an intuitionistic fuzzy dense ideal of R.

Definition (4.25) If $A \in IFS(R)$. Then the intuitionistic fuzzy ideal of the form ann(A) is called an intuitionistic fuzzy ideal. Thus if A is an intuitionistic fuzzy annihilator ideal if and only if A = ann(B) for some $B \in IFS(R)$ with with $\mu_B(0) = 1$, $\nu_B(0) = 0$.
Remark (4.26) In view of theorem (4.19) it follows that A is an annihilator ideal of R implies $ann(ann(A)) = A$.

Theorem (4.27) The annihilator ideals in a semiprime ring form a complete Boolean algebra with intersection as infimum and ann as complementation.

Proof. Since $\bigcap_{i \in I} ann(A_i) = ann\left(\sum_{i \in I} A_i\right)$, so any intersection of annihilator ideals is an intuitionistic fuzzy annihilator ideal. Hence these ideals form a complete semi-lattice with intersection as infimum. To show that they form a Boolean algebra it remain to show that:
$A \cap ann(B) = \chi_0$ if and only if $A \subseteq B$, for any annihilator ideals A and B.
If $A \subseteq B$ then $A \cap ann(B) \subseteq B \cap ann(B) = \chi_0$.
Conversary, let $A \cap ann(B) = \chi_0$.
Now, $A\, ann(B) \subseteq A \cap ann(B) = \chi_0 \Rightarrow A \subseteq ann(ann(B)) = B$.

Theorem (4.28) Let M be a non-zero R-module. Suppose that there exist no ideal A maximal among the annihilators of non-zero intuitionistic fuzzy submodules (IFSMs) of M. Then A is an intuitionistic fuzzy prime ideal of R.

Proof. Since A is maximal among the annihilators of non-zero intuitionistic fuzzy submodules (IFSMs) of M. Therefore there is an IFSM B ($\neq \chi_0$) of M such that $A = ann(B)$.

Suppose $P, Q \in IFI(R)$ properly containing A (i.e., $A \subset P$ and $A \subset Q$) such that $PQ \subseteq A$.
If $QB = \chi_0$ then $Q \subseteq ann(B) = A$, which is a contradiction to our supposition so $QB \neq \chi_0$.
Now, $PQ \subseteq A \Rightarrow P(QB) \subseteq AB = ann(B)B = QB = \chi_0$. So $Q \subseteq ann(QB)$.

Hence $A \subseteq ann(QB)$. This is a contradiction of the maximality of A. So A is an intuitionistic fuzzy prime ideal of R.

Remark (4.29) If $A \in IFM(M)$, $A \neq \chi_\theta$ satisfying one (hence both) the condition of Theorem (4.11) then A is called an intuitionistic fuzzy prime submodule of M.

Theorem (4.30) If A is an intuitionistic fuzzy prime submodule of M then ann(A) is an intuitionistic fuzzy prime ideal of R.

Proof. Let A be an intuitionistic fuzzy prime submodule of M and $PQ \subseteq ann(A)$, where Q is not contained in ann(A). Then $\chi_\theta \neq QA \subseteq A$. Now $PQ \subseteq ann(A) \Rightarrow (PQ)A \subseteq ann(A)A = \chi_\theta$.
So, $P \subseteq ann(QA) = ann(A)$, as A is prime. Hence ann(A) is prime ideal of R.

5. CONCLUSIONS

In this paper we have developed the concept of residual quotient of intuitionistic fuzzy subset of rings and modules and then study some results on annihilator of an intuitionistic fuzzy subset of a R-module. Using this notion, we investigate some important characterization of intuitionistic fuzzy annihilator of subsets of modules. The annihilator of union (sum) of intuitionistic fuzzy submodules are obtained. Annihilator of intuitionistic fuzzy ideal of prime ring, semi prime ring are also obtained. Using the concept of intuitionistic fuzzy annihilators, intuitionistic fuzzy prime submodules and intuitionistic fuzzy ideals are defined and various related properties are established.

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