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Resonance in the Motion of Geocentric Satellites due to Poynting-Robertson Drag

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ABSTRACT:

The problem of resonance in a geocentric synchronous satellite under the gravitational forces of the Sun and the Earth subject to Poynting-Robertson (P-R) drag is the subject matter of this paper. Based on the assumption that the two bodies the Earth and the Sun lie in ecliptic plane and the satellite in the orbital plane. Five resonance points results from commensurability between the mean motion of the satellite and the average angular velocity of the Earth. Out of all resonance, the 3:2 and 1:2 resonance occurs only due to velocity dependent terms of P-R drag. We have determined the amplitude and time period of the oscillation in two different cases at those resonance points.

KEYWORDS:

Three-body problem; Ecliptic plane; Resonance; Poynting-Robertson drag; Geocentric satellites

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1. Introduction

Poynting [1] and Robertson [2] have investigated the radiation pressure, the Doppler shift of the incident radiation and the Poynting drag which generally constitutes the radiation force on a particle exerted by a radiating body. Bhatnagar and Gupta [3] examined resonance caused by solar radiation pressure in the motion of an artificial Earth's satellite. Hamiltonian and the generating function, which is dependent on solar radiation pressure was expanded in the power series of a small parameter. Klacka [5] surveyed the problem of the action of the solar radiation on the motion of interplanetary dust particles and he explained the difference between the action of electromagnetic solar radiation and that of solar wind. They differ not only from the point of view of physical nature of these phenomena but also from the point of view of dust particle's orbital evolution. Ragos et al [6] numerically studied the existence of equilibrium points for particles and their stability, moving in the vicinity of two massive bodies which exert light radiation pressure.

Ragos et al [7] have discussed the photo gravitational circular restricted three-body problem including the P-R effect to describe the effect in the vicinity of two massive radiating bodies. A modified bisection method is used to compute the position of the equilibrium and thereby establishing the stability. Liou and Zook [8] have explored the effect of radiation pressure, P-R drag, and solar wind drag on the dust grains trapped in the mean motion resonances with the Sun and Jupiter in the restricted three-body problem having negligible dust mass. They especially examined the evolution of dust grain in the 1:1 resonances. Kushvah [9] has investigated effect of P-R drag on linear stability of equilibrium points in the generalized photo gravitational Chermnykh's problem when a bigger primary is radiating and a smaller primary is an oblate spheroid. It is found that when P-R effect is taken into account, these points are unstable in a linear sense.

Lhotka et al [10] surveyed the stability of motion to the Lagrangian equilibrium points L4 and L5 in the framework of restricted three-body problem, along with the elliptic and spatial, subject to the radial component of P-R drag. Yadav and Aggarwal [11-13] in the series of papers discussed the resonances in a geo-centric synchronous satellite under the gravitational forces of the Moon, the Sun and the Earth including its equatorial elasticity. The amplitude and the time period of the oscillation was determined by using the procedure of Brown and Shook [4]. Jain and Aggarwal [14] investigated the existence of non-collinear liberation points and their stability in the circular restricted three-body problem in which they considered the smaller primary as an oblate spheroid and bigger one a point mass including the effect of dissipative forces specially Stokes drag. Pushparaj and Sharma [15] used the method of Poincare surface of section to study interior resonance periodic orbits around the Sun in the Sun-Jupiter photo gravitational restricted three-body problem.

The period of time for these orbits is found to decease with the increase in the Sun's radiation pressure. Most of the writers have explored two of the three; P-R drag, three-body problem or resonance. By taking into consideration all the three factors, we have attempted to fill the above said gap. The intent of this paper is to investigate the resonance in the motion of geocentric synchronous satellite under P-R drag of the three-body problem. Careful analysis of equations of motion in Sect. 2 of this paper disclose that there are five points $R_i's(i=1-5)$ of resonance in the motion of the orbiting satellite commensurable between *n* and $\dot\phi$ where *n* is the mean motion of the satellite and $\dot\phi$ the average angular velocity of the Earth. Appraisal of the corresponding amplitudes and time periods at resonance points have been done in section 3. Section 4 scrutinize the difference of amplitude and time period with respect to ϕ and different values of *q*.

2. Equation of motion

Let S represents the Sun, E the Earth and P the satellite with their masses M_S, M_E and M_P respectively. The satellite moves around the Earth in ecliptic plane. Let X, Y and Z be the co-ordinate system with origin at the center of the Earth and unit vectors $\hat I, \hat J$ and $\hat K$ along the coordinate's axes. Let X_0, Y_0 and Z_0 be another set of co-ordinate system in the same plane with origin at the center of the Earth, and with unit vectors $\hat I_0, \hat J_0$ and $\hat K_0$ along the co-ordinate axes respectively (1b). Let the satellite be revolving about the Earth with angular velocity $\vec\omega$ and the system is also revolving with the same angular velocity $\vec\omega$. Let $\overrightarrow{SE} = \vec r_E$, $\overrightarrow{SP} = \vec r_S$, $\overrightarrow{EP} = \vec r$, where S represents the Sun. γ = vernal equinox. θ = the angle between direction of ascending and direction of the satellite. $\phi =$ the angle between direction of ascending and direction of the Sun. $c =$ velocity of light.

Let $\vec F_P$ be the P-R drag per unit mass acting on the satellite due to radiating body (Sun) in the arbitrary direction as shown in Fig. (1a), given by [6],

$$M_P \vec F_P = \vec f_1 + \vec f_2 + \vec f_3,$$

Where $\vec f_1 = F\,\vec r_S / r_S =$ (Radiation pressure),

$\vec f_2 = -F\dfrac{(\vec v\cdot\vec r_S)}{c}\dfrac{\vec r_S}{r_S} =$ (Doppler shift owing to the motion), $\vec f_3 = -F\,\vec v/c$ (Force due to the absorption and re-emission of part of the incident radiation), $\vec v =$ velocity of *P*. *F* = the measure of the radiation pressure. $G =$ Gravitational constant. The relative motion of the satellite with respect to the Earth is obtained by,

$$\ddot{\vec r} = \ddot{\vec r}_S - \ddot{\vec r}_E = \frac{\vec F_{SP} + \vec F_{EP} + \vec F_P M_P}{M_P} - \frac{\vec F_{SE}}{M_E},$$

Where $\vec F_{SP} = -G\dfrac{M_S M_P}{r_S^{\,3}}\vec r_S$, $\vec F_{SE} = -G\dfrac{M_S M_E}{r_E^{\,3}}\vec r_E$,

$\vec F_{EP} = -G\dfrac{M_P M_E}{r^3}\vec r$. Thus,

$$\ddot{\vec r} = -qF_g\frac{\vec r_S}{r_S} - G\frac{M_E}{r^3}\vec r + G\frac{M_s}{r_E^{\,3}}\vec r_E + (1-q)F_g\left\{\frac{(\vec v.\vec r_S)}{c}\frac{\vec r_S}{r_S} - \frac{\vec v}{c}\right\},$$

Where $q = 1 - \dfrac{F_p}{F_g}$, $F_g = \dfrac{GM_S}{r_s^2}$, $q = 1 - p$, $p = \dfrac{F_p}{F_g}$.

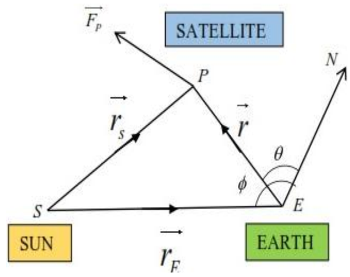

Fig. 1(a): Configuration of 3-body problem without co-ordinate

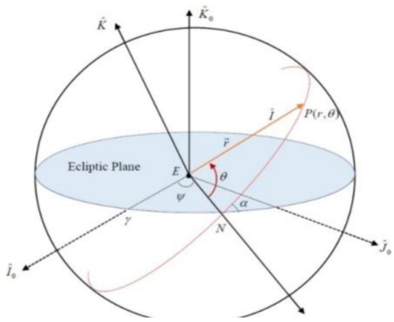

Fig. 1(b): Configuration of 3-body problem with co-ordinate

Motion of the Earth relative to the Sun is given by, $\dot\phi^2 = GM_S / r_E^{\,3}$. Also $\vec r = r\hat I$, $\hat I$ is the unit vector along $\vec r$, $\vec r_E = r_E \hat r_E$, $\hat r_E = \mathrm{Cos}\,\phi\,\hat I_0 + \mathrm{Sin}\,\phi\,\hat J_0$, $\vec r_E = r_E\,\mathrm{Cos}\,\phi\,\hat I_0 + r_E\,\mathrm{Sin}\,\phi\,\hat J_0$. Using these values in the equation of motion of the satellite with respect to the Earth in vector form can be written as

$$\ddot{\vec r} = \dot\phi^2 r_E(\mathrm{Cos}\,\phi\,\hat I_0 + \mathrm{Sin}\,\phi\,\hat J_0) - G\frac{M_E}{r^3}r\hat I - (1-q)F_g \left\{\frac{(\vec v.\vec r_S)}{c}\frac{\vec r_S}{r_S} - \frac{\vec v}{c}\right\} - \frac{qGM_S\vec r_S}{r_S}. \qquad (1)$$

In the rotating frame of reference with angular velocity $\vec\omega$ of the satellite about the center of the Earth, we have

$$\ddot r = \frac{\partial^2 r}{\partial t^2}\hat I + 2\frac{\partial r}{\partial t}\left(\vec\omega\times\hat I\right) + r\left(\frac{\partial\vec\omega}{\partial t}\times\hat I\right) + r\left\{\begin{matrix}\left(\vec\omega\cdot\hat I\right)\vec\omega \\ -\left(\vec\omega\cdot\vec\omega\right)\hat I\end{matrix}\right\}, \qquad (2)$$

Where $\vec\omega = \dot\theta\hat K$. Taking dot products of Eqns. (1) and (2) with $\hat I, \hat J$ respectively and equating the respective coefficients, we get the equations of motion of the satellite in the synodic coordinate system (see Table 1).

Table 1: Relationship between coordinate system

	$\hat I_0$	$\hat J_0$	$\hat K_0$
I	a_x	b_x	c_x
J	a_y	b_y	c_y
K	a_z	b_z	c_z

Eqns. (3) and (4) are the required equations of motion of the satellite in polar form. These Equations are not integrable, therefore we follow the perturbation technique and replace $r,\alpha,\dot{\theta}$ and $\dot{\psi}$ by their steady state values r_0, α_0, $\dot{\theta}_0$ and $\dot{\psi}_0$ respectively. We take $\theta=\dot{\theta}_0 t,\ \psi=\dot{\psi}_0 t$.

$$\ddot{r}-r\dot{\theta}^2+\frac{GM_E}{r^2}=-\frac{qGM_S(\vec{r}_S\cdot\hat{I})}{r_S^3}+\dot{\phi}^2(cos\theta\cos(\phi-\psi)-cos\alpha\, sin\theta\, sin(\phi-\psi))-(1-q)\frac{GM_S}{r_S^2}\left\{\frac{(\vec{v}\cdot\vec{r}_S)(\vec{r}_S\cdot\hat{I})}{cr_S}+\frac{(\vec{v}\cdot\hat{I})}{c}\right\} \tag{3}$$

$$\frac{d\left(r^2\dot{\theta}\right)}{dt}=\frac{-qGM_S r(\vec{r}_S\cdot\hat{J})}{r_S^3}-\dot{\phi}^2 r\begin{Bmatrix}sin\theta\cos(\phi-\psi)\\-cos\alpha cos\theta\, sin(\phi-\psi)\end{Bmatrix}-(1-q)\frac{GM_S r}{r_S^2}\left\{\frac{(\vec{v}\cdot\vec{r}_S)(\vec{r}_S\cdot\hat{J})}{cr_S}+\frac{(\vec{v}\cdot\hat{J})}{c}\right\} \tag{4}$$

Putting the steady state values in the R.H.S of Eqns. (3) and (4), we get,

$$\ddot{r}-r\dot{\theta}^2+\frac{GM_E}{r^2}=\frac{-qGM_S(\vec{r}_S\cdot\hat{I})}{r_S^3}+\dot{\phi}^2\begin{Bmatrix}cos\dot{\theta}_0 t\cos\left(\dot{\phi}-\dot{\psi}_0\right)t\\-cos\alpha\, sin\dot{\theta}_0 t\, sin\left(\dot{\phi}-\dot{\psi}_0\right)t\end{Bmatrix}-(1-q)\frac{GM_S}{r_S^2}\left\{\frac{(\vec{v}\cdot\vec{r}_S)(\vec{r}_S\cdot\hat{I})}{cr_S}+\frac{(\vec{v}\cdot\hat{I})}{c}\right\} \tag{5}$$

$$\frac{d\left(r^2\dot{\theta}\right)}{dt}=\frac{-qGM_S r_0(\vec{r}_S\cdot\hat{J})}{r_S^3}+\dot{\phi}^2 r_0\begin{Bmatrix}sin\dot{\theta}_0 t\cos\left(\dot{\phi}-\dot{\psi}_0\right)t\\-cos\alpha_0\, cos\dot{\theta}_0 t\, sin\left(\dot{\phi}-\dot{\psi}_0\right)t\end{Bmatrix}-(1-q)\frac{GM_S r_0}{r_S^2}\left\{\frac{(\vec{v}\cdot\vec{r}_S)(\vec{r}_S\cdot\hat{J})}{cr_S}+\frac{(\vec{v}\cdot\hat{J})}{c}\right\}, \tag{6}$$

Also as $e<1$, we have $\left(1+e\cos nt\right)^{h1}\approx 1+h_1 e\cos nt$. Where

$a_x=cos\theta\, cos\psi-cos\alpha\, sin\theta\, sin\psi,\ a_y=-sin\theta\, cos\psi-cos\alpha\, cos\theta\, sin\psi,\ a_z=sin\alpha\, sin\psi,\ b_x=cos\theta\, sin\psi+cos\alpha\, sin\theta\, cos\psi$,

$b_y=-sin\theta\, sin\psi+cos\alpha\, cos\theta\, cos\psi b_z=-cos\psi\, sin\alpha,\ c_x=sin\alpha\, sin\theta,\ c_y=sin\alpha\, cos\theta,\ c_z=cos\alpha$

$$\vec{v}=\left\{-r_E\left(\dot{\phi}-\dot{\psi}_0\right)cos\dot{\theta}_0 t\, sin\left(\dot{\phi}-\dot{\psi}_0\right)t+r_E\left(\dot{\phi}-\dot{\psi}_0\right)sin\dot{\theta}_0 t\cos\left(\dot{\phi}-\dot{\psi}_0\right)t\cos\alpha_0\right\}\hat{I}+\left\{r_0\dot{\theta}+r_E\left(\dot{\phi}-\dot{\psi}_0\right)sin\dot{\theta}_0 t\, sin\left(\dot{\phi}-\dot{\psi}_0\right)t\right.$$
$$\left.+r_E\left(\dot{\phi}-\dot{\psi}_0\right)cos\dot{\theta}_0 t\cos\left(\dot{\phi}-\dot{\psi}_0\right)t\cos\alpha_0\right\}\hat{J}+\left\{-r_E\left(\dot{\phi}-\dot{\psi}_0\right)sin\alpha_0 cos\left(\dot{\phi}-\dot{\psi}_0\right)t\right\}\hat{K}$$

Taking $r^2\dot{\theta}$ = constant, $r=1/u$ and using above we get,

$$\frac{d^2u}{dt^2}+n^2u=\frac{GM_E}{r^4{}_0}-\frac{\dot{\phi}^2 r_E}{2a^2\left(1-e^2\right)^2}\left\{\cos\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+\cos\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t+e\cos\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+2e\cos\left(\dot{\phi}-\dot{\psi}_0\right)t\right.$$
$$\left.+e\cos\left(2\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t\right\}-\frac{\dot{\phi}^2\cos\alpha_0 r_E}{2a^2\left(1-e^2\right)^2}\left\{\cos\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t-e\cos\left(\dot{\phi}-\dot{\psi}_0\right)t+e\cos\left(\dot{\phi}-\dot{\psi}_0\right)t+e\cos\left(2\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t\right.$$
$$\left.-e\cos\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-\cos\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t\right\}-\frac{qGM_s\left(1+e\cos\dot{\theta}_0 t\right)}{r_s^3a^2\left(1-e^2\right)}+\frac{qGM_s r_E}{2r_s^3a^2\left(1-e^2\right)}\left\{\cos\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t\right.$$
$$+e\cos\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+2e\cos\left(\dot{\phi}+\dot{\psi}_0\right)t+\cos\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t+e\cos\left(2\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t+\cos\alpha_0\cos\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t$$
$$\left.+e\cos\alpha_0\cos\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-\cos\alpha_0\cos\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-e\cos\alpha_0\cos 2\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t\right\}+\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)r_E}{2ca^2\left(1-e^2\right)^2}$$
$$\left\{\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t-e\sin\left(2\dot{\theta}_0-\dot{\phi}-\dot{\psi}_0\right)t+e\sin\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+2c\sin\left(\dot{\phi}-\dot{\psi}_0\right)t\right\}$$
$$+\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)\cos\alpha_0}{2ca^2\left(1-e^2\right)^2}\left\{\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t+e\sin\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+e\sin\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t\right\}$$
$$+\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)\left(\cos^2\alpha_0-1\right)r_E{}^3}{8r_sca^2\left(1-e^2\right)^2}\left\{\sin\left(\dot{\theta}_0+3\dot{\phi}-3\dot{\psi}_0\right)t-\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t+\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-\sin\left(\dot{\theta}_0-3\dot{\phi}+3\dot{\psi}_0\right)t\right.$$
$$+e\sin\left(2\dot{\theta}_0+3\dot{\phi}-3\dot{\psi}_0\right)t+2e\sin 3\left(\dot{\phi}-\dot{\psi}_0\right)t-e\sin\left(2\dot{\theta}_0-\dot{\phi}+3\dot{\psi}_0\right)t+2e\sin\left(\dot{\phi}-\dot{\psi}_0\right)t+e\sin\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t$$
$$\left.-e\sin\left(2\dot{\theta}_0-3\dot{\phi}-3\dot{\psi}_0\right)t\right\}+\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)r_E^3\left(\cos^2\alpha_0-1\right)\cos^2\alpha_0}{8r_sca^2\left(1-e^2\right)^2}\left\{\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-\sin\left(\dot{\theta}_0+3\dot{\phi}-3\dot{\psi}_0\right)t+\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t\right.$$
$$-\sin\left(\dot{\theta}_0-3\dot{\phi}+3\dot{\psi}_0\right)t-e\sin\left(2\dot{\theta}_0-3\dot{\phi}+3\dot{\psi}_0\right)t+e\sin\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+2e\sin\left(\dot{\phi}-\dot{\psi}_0\right)t+e\sin\left(2\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t$$

$$-e\sin\left(2\dot{\theta}_0+3\dot{\phi}-3\dot{\psi}_0\right)t\Big\}+\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)r_E^3\left(\cos^2\alpha_0-1\right)}{2r_sca\left(1-e^2\right)}\Big\{\sin 2\left(\dot{\phi}-\dot{\psi}_0\right)t+\frac{e}{2}\sin\left(\dot{\theta}_0+2\dot{\phi}-\dot{\psi}_0\right)t-\frac{e}{2}e\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t\Big\}$$

$$+\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)r_E^2\cos^2\alpha_0}{4r_sca\left(1-e^2\right)}\Big\{\sin\left(2\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t+\sin\left(2\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t+\frac{e}{2}\sin\left(3\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t+\frac{e}{2}\sin\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t$$

$$+\frac{e}{2}\sin\left(3\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t+\frac{e}{2}\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t\Big\}-\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)r_E^2}{2r_sca\left(1-e^2\right)}\Big\{\frac{1}{2}\sin 2\left(\dot{\phi}-\dot{\psi}_0\right)t+\frac{1}{4}\sin\left(2\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t$$

$$+\frac{1}{4}\sin\left(2\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t+\frac{e}{4}\sin\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t-\frac{e}{4}\sin\left(\dot{\theta}_0-2\dot{\phi}-2\dot{\psi}_0\right)t+\frac{e}{8}\sin\left(3\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t+\frac{e}{8}\sin\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t$$

$$+\frac{pF_g\dot{\theta}_0\cos\alpha_0 r_E}{2r_sc}\Big\{\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t\Big\}+\frac{pr_E^3\sin^2\alpha_0\left(\dot{\phi}-\dot{\psi}_0\right)}{8r_sca^2\left(1-e^2\right)^2}\Big\{\sin\left(\dot{\theta}_0+3\dot{\phi}-3\dot{\psi}_0\right)t+\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t$$

$$-\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t-\sin\left(\dot{\theta}_0-3\dot{\phi}+3\dot{\psi}_0\right)t+e\sin\left(2\dot{\theta}_0+3\dot{\phi}-3\dot{\psi}_0\right)t+e\sin 3\left(\dot{\phi}-\dot{\psi}_0\right)t+e\sin\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+2e\sin\left(\dot{\phi}-\dot{\psi}_0\right)t$$

$$-e\sin\left(2\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t-e\sin\left(\dot{\theta}_0-3\dot{\phi}+3\dot{\psi}_0\right)t\Big\}+\frac{pF_gr_E^3\sin^2\alpha_0\left(\dot{\phi}-\dot{\psi}_0\right)\cos\alpha_0}{8r_sca^2\left(1-e^2\right)^2}\Big\{\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t-\sin\left(\dot{\theta}_0-3\dot{\phi}+3\dot{\psi}_0\right)t$$

$$-\sin\left(\dot{\theta}_0+3\dot{\phi}-3\dot{\psi}_0\right)t+\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+e\sin\left(2\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t-e\sin\left(2\dot{\theta}_0-3\dot{\phi}+3\dot{\psi}_0\right)t-e\sin\left(2\dot{\theta}_0+3\dot{\phi}-3\dot{\psi}_0\right)t$$

$$+e\sin\left(2\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t\Big\}+\frac{pF_gr_E^3\sin^2\alpha_0\left(\dot{\phi}-\dot{\psi}_0\right)}{2r_sca\left(1-e^2\right)}\Big\{\sin 2\left(\dot{\phi}-\dot{\psi}_0\right)t+\frac{e}{2}\sin\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t-\frac{e}{2}\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t\Big\}$$

$$-\frac{e}{8}\sin\left(3\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t-\frac{e}{8}\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t\Big\}-\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)r_E}{2r_sc}\Big\{\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t\Big\}$$

$$+\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)r_E^2\cos^2\alpha_0}{4r_sca\left(1-e^2\right)}\Big\{\sin 2\left(\dot{\phi}-\dot{\psi}_0\right)t-\frac{1}{2}\sin 2\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+\frac{1}{2}\sin 2\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t+\frac{e}{2}\sin\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t$$

$$-\frac{e}{2}\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t-\frac{e}{2}\sin\left(3\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t-\frac{e}{4}\sin\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t+\frac{e}{4}\sin\left(3\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t+\frac{e}{4}\sin\left(\dot{\theta}_0-2\dot{\phi}+\dot{\psi}_0\right)t\Big\}$$

$$+\frac{pF_g\left(\dot{\phi}-\dot{\psi}_0\right)r_E\cos\alpha_0}{2r_sc}\Big\{\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t\Big\}-\frac{pF_g\dot{\theta}_0r_E^2}{4r_sca\left(1-e^2\right)}\Big\{\sin 2\dot{\theta}_0t+\frac{1}{2}\sin 2\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t$$

$$+\frac{1}{2}\sin 2\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t+\frac{e}{2}\sin 3\dot{\theta}_0t+\frac{e}{2}\sin\dot{\theta}_0t+\frac{e}{4}\sin\left(3\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t+\frac{e}{4}\sin\left(\dot{\theta}_0+2\dot{\phi}-\dot{\psi}_0\right)t+\frac{e}{4}\sin\left(3\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t$$

$$+\frac{e}{4}\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t\Big\}-\frac{pF_g\dot{\theta}_0r_E^2\cos\alpha_0}{4r_sca\left(1-e^2\right)}\Big\{\sin 2\left(\dot{\phi}-\dot{\psi}_0\right)t-0.5\sin 2\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+0.5\sin 2\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t$$

$$+0.5e\,\text{in}\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t-0.5e\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t-0.25e\sin\left(3\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t-0.25e\sin\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t$$

$$+\frac{e}{4}\sin\left(3\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t+\frac{e}{4}\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t\Big\}-\frac{pF_g\dot{\theta}_0r_E}{2r_sc}\Big\{\sin\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t+\sin\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t\Big\}$$

$$+\frac{pF_gr_E^2\cos^2\alpha_0}{4r_sca\left(1-e^2\right)}\Big\{\sin 2\dot{\theta}_0t-0.5\sin 2\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-0.5\sin 2\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t+\frac{e}{2}\sin 3\dot{\theta}_0t+\frac{e}{2}\sin\dot{\theta}_0t$$

$$-\frac{e}{4}\sin\left(3\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t-\frac{e}{4}\sin\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t-\frac{e}{4}\sin\left(3\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t-\frac{e}{4}\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t\Big\}+\frac{pF_g\dot{\theta}_0r_E^2\cos\alpha_0}{4r_sca\left(1-e^2\right)}$$

$$\Big\{\sin 2\left(\dot{\phi}-\dot{\psi}_0\right)t+0.5\sin 2\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)t-0.5\sin 2\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)t+0.25e\sin\left(3\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t$$

$$+0.75e\sin\left(\dot{\theta}_0+2\dot{\phi}-2\dot{\psi}_0\right)t-0.25e\sin\left(3\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t-0.75e\sin\left(\dot{\theta}_0-2\dot{\phi}+2\dot{\psi}_0\right)t\Big\}$$

Since the solution of unperturbed system, $d^2u/d\theta^2+u=GM_E/r_0^2\dot{\theta}_0^2$, is given by $l/r=1+e\cos(\theta-\omega)$, Where, $r^2\dot{\theta}=\text{constant}, l=a\left(1-e^2\right)$, and $e,\omega=$ constant of integrations.

$$u=\frac{1+e\cos(\theta-\omega)}{a\left(1-e^2\right)},\quad \theta-\omega=f=\dot{\theta}_0t=nt\ \ \text{(say)}.$$

Thus we can write,

$$\frac{d^2u}{d\theta^2}+n^2u=C_1+C_2sinnt+C_3sin2nt+C_4sin3nt+C_5\cos\left(\dot{\phi}-\psi_0\right)t+C_6\sin\left(\dot{\phi}-\psi_0\right)t+C_7\cos\left(n+\dot{\phi}-\psi_0\right)t+C_8\sin\left(n+\dot{\phi}-\psi_0\right)t$$
$$+C_9\cos\left(n-\dot{\phi}+\psi_0\right)t+C_9\cos\left(n-\dot{\phi}+\psi_0\right)t+C_{10}\sin\left(n-\dot{\phi}+\psi_0\right)t+C_{11}\cos\left(2n+\dot{\phi}-\psi_0\right)t+C_{12}\sin\left(2n+\dot{\phi}-\psi_0\right)t \quad (7)$$
$$+C_{13}\cos\left(2n-\dot{\phi}+\psi_0\right)t+C_{14}\sin\left(2n-\dot{\phi}+\psi_0\right)tC_{15}\sin 2\left(\dot{\phi}-\dot{\phi}_0\right)t+C_{16}\sin\left(2n+2\dot{\phi}-2\psi_0\right)t+C_{17}\sin\left(2n-2\dot{\phi}+2\dot{\psi}_0\right)t$$
$$+C_{18}\sin\left(n+2\dot{\phi}-2\dot{\psi}_0\right)t+C_{19}\sin\left(n-2\dot{\phi}+2\dot{\psi}_0\right)t+C_{20}\sin\left(3n+2\dot{\phi}-2\dot{\psi}_0\right)t+C_{21}\sin\left(3n-2\dot{\phi}+2\dot{\psi}_0\right)t+C_{22}\cos nt$$

The solution is given by

$$u=ACos\left(nt-\alpha\right)+\frac{C_1}{n^2}+\frac{C_2tcost}{2n}+\left(\frac{C_3sin2nt}{n^2-\left(2n\right)^2}\right)+\left(\frac{C_4sin3nt}{n^2-\left(3n\right)^2}\right)+\left(\frac{C_5cos\left(\dot{\phi}-\dot{\psi}_0\right)t}{n^2-\left(\dot{\phi}-\dot{\psi}_0\right)^2}\right)+\left(\frac{C_6sin\left(\dot{\phi}-\dot{\psi}_0\right)t}{n^2-\left(\dot{\phi}-\dot{\psi}_0\right)^2}\right)$$
$$+\left(\frac{C_7cos\left(n+\dot{\phi}-\dot{\psi}_0\right)t}{n^2-\left(n+\dot{\phi}-\dot{\psi}_0\right)^2}\right)+\left(\frac{C_8sin\left(n+\dot{\phi}-\dot{\psi}_0\right)t}{n^2-\left(n+\dot{\phi}-\psi_0\right)^2}\right)+\left(\frac{C_9cos\left(n-\dot{\phi}+\dot{\psi}_0\right)t}{n^2-\left(n-\dot{\phi}+\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{10}sin\left(n-\dot{\phi}+\psi_0\right)t}{n^2-\left(n-\dot{\phi}+\dot{\psi}_0\right)^2}\right)$$
$$+\left(\frac{C_{11}cos\left(2n+\dot{\phi}-\dot{\psi}_0\right)t}{n^2-\left(2n+\dot{\phi}-\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{12}sin\left(2n+\dot{\phi}-\dot{\psi}_0\right)t}{n^2-\left(2n+\dot{\phi}-\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{13}cos\left(2n-\dot{\phi}+\dot{\psi}_0\right)t}{n^2-\left(2n-\dot{\phi}+\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{14}sin\left(2n-\dot{\phi}+\dot{\psi}_0\right)t}{n^2-\left(2n-\dot{\phi}+\dot{\psi}_0\right)^2}\right) \quad (8)$$
$$+\left(\frac{C_{15}sin2\left(\dot{\phi}-\dot{\psi}_0\right)t}{n^2-4\left(\dot{\phi}-\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{16}sin\left(2n+2\dot{\phi}-2\dot{\psi}_0\right)t}{n^2-\left(2n+2\dot{\phi}-2\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{17}cos\left(2n-2\dot{\phi}+2\dot{\psi}_0\right)t}{n^2-\left(2n-2\dot{\phi}+2\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{18}sin\left(n+2\dot{\phi}-2\dot{\psi}_0\right)t}{n^2-\left(n+2\dot{\phi}-2\dot{\psi}_0\right)^2}\right)$$
$$+\left(\frac{C_{19}sin\left(n-2\dot{\phi}+2\dot{\psi}_0\right)t}{n^2-\left(n-2\dot{\phi}+2\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{20}sin\left(3n+2\dot{\phi}-2\dot{\psi}_0\right)t}{n^2-\left(3n+2\dot{\phi}-2\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{21}sin\left(3n-2\dot{\phi}+2\dot{\psi}_0\right)t}{n^2-\left(3n-2\dot{\phi}+2\dot{\psi}_0\right)^2}\right)+\left(\frac{C_{22}t\cos nt}{2n}\right)$$

Where C_i's are given in Appendix A.

2.1. Resonance

It is clear that the motion become indeterminate if any one of the denominator vanishes in Eqn. (7), and hence the resonance occur at those points. It is found that resonance occurs at five point $n=\dot{\phi}$, $n=2\dot{\phi}$, $3n=\dot{\phi}$, $2n=\dot{\phi}$, $3n=2\dot{\phi}$. Out of all resonance, the 3:2 and 1:2 resonance occurs only due to P-R drag. Amplitude and time periods at resonance points are deduced below.

3. Time period and amplitude at the resonance point

To determine the amplitude and time period at the resonance points. We have followed the special method [3]. Resonance at $n=\dot{\phi}$. In our problem solution of Eqn. (7) is periodic and known which is the condition of Ref. [3]. So we followed the same to determine amplitude and time period at $n=\dot{\phi}$ It is suggested to obtain the solution of Eqn. (7) when that of

$$\frac{d^2u}{dt^2}+n^2u=0 \quad (9)$$

is periodic and is known. The solution of Eqn. (9) is

$$u=k\,\text{Cos}\;s,$$

Where

$$s=nt+\varepsilon,\quad n=\sqrt{k_1}/k=\text{Function of }k; \quad (10)$$

K, k_1 and ε are arbitrary constants.

As we are probing the resonance in the motion of the satellite at the point $n=\dot{\phi}$, the resulting Eqn. (7) can be written as

$$\frac{d^2u}{dt^2}+n^2u=HA'\,\text{Cos}\,n't=H\psi' \text{ (Say)},$$

Where

$$H=\frac{\left(\dot{\phi}^2r_s^{\,3}-qGM_s\right)r_Ee}{a^2\left(1-e^2\right)^2r_s^{\,3}}=\text{Constant},$$

$$A'=-1, \quad (11)$$

$$\psi'=\frac{\partial\psi}{\partial u}=A'\,\text{Cos}\,n't,\quad \psi=A'\,\text{Cos}\,n't,$$

$$\psi=\frac{kA'}{2}\left\{\text{Cos}\left(n't+s\right)+\text{Cos}\left(n't-s\right)\right\}.$$

Then

$$\frac{dk}{dt}=\frac{H}{W}\frac{\partial u}{\partial s}\psi'=\frac{H}{W}\frac{\partial\psi}{\partial s}, \quad (12)$$

$$\frac{ds}{dt}=n-\frac{H}{W}\frac{\partial u}{\partial k}\psi'=n-\frac{H}{W}\frac{\partial\psi}{\partial k}, \quad (13)$$

Where

$$W=\frac{\partial}{\partial k}\left(n\frac{\partial u}{\partial s}\right)\frac{\partial u}{\partial s}-n\frac{\partial^2u}{\partial s^2}\frac{\partial u}{\partial k},$$

= a function of k only. (14)

Since n, W are function of k only, we can put Eqns. (12) and (13) into canonical form with new variables k_1 *and* B defined by

$$dk_1 = W dk, \tag{15}$$
$$dB = -n dk_1 = -nW dk, \tag{16}$$

Eqns. (15) and (16) can be put in the form

$$\frac{dk_1}{dt} = \frac{\partial}{\partial s}(B + H\psi), \quad \frac{ds}{dt} = -\frac{\partial}{\partial s}(B + H\psi).$$

Differentiating Eqn. (13) with respect to t and substituting the expression for $\frac{ds}{dt}$ *and* $\frac{dk}{dt}$, we have

$$\frac{d^2 s}{dt^2} = \frac{H}{W}\left\{\frac{\partial n}{\partial k}\frac{\partial \psi}{\partial s} - n\frac{\partial^2 \psi}{\partial s \partial k} - \frac{\partial^2 \psi}{\partial k \partial s}\right\} +$$
$$\frac{H^2}{K^2}\left\{\frac{\partial^2 \psi}{\partial s \partial k}\frac{\partial \psi}{\partial k} - W\frac{\partial}{\partial k}\left(\frac{1}{W}\frac{\partial \psi}{\partial k}\right)\frac{\partial \psi}{\partial s}\right\} \tag{17}$$

Since the last expression of Eqn. (17) has the factor H^2 it may, in general, be neglected in a first approximation. In Eqn. (11) we find s and t are present in ψ' as sum of the periodic terms with argument $s' = s - n't$. In our case, the affected term is,

$$\psi = \frac{k}{2} A' \operatorname{Cos} s'. \tag{18}$$

Eqn. (17) for s' is then,

$$\frac{d^2 s'}{dt^2} + (n - n')^2 \frac{H}{W}\left\{\frac{\partial}{\partial q}\left(\frac{1}{n - n'}\frac{\partial \psi}{\partial s'}\right)\right\} = 0,$$

or

$$\frac{d^2 s'}{dt^2} + (n - n')^2 \frac{H}{W}\left\{\frac{\partial}{\partial q}\left(\frac{qA'}{n - n'}\right)\right\} \operatorname{Sin} s' = 0 \tag{19}$$

At first approximation, we put $k = k_0, n = n_0, W = W_0$. Then Eqn. (19) can be written as,

$$\frac{d^2 s'}{dt^2} + (n_0 - n')^2 \frac{H}{W_0}\left\{\frac{\partial}{\partial k}\left(\frac{kA'}{n - n'}\right)\right\}_0 \operatorname{Sin} s' = 0 \tag{20}$$

If the oscillation be small, Eqn. (20) can be put in the form,

$$\frac{d^2 \dot{s}}{dt^2} + (n_0 - n')^2 \frac{H}{W_0}\left\{\frac{\partial}{\partial k}\left(\frac{kA'}{n - n'}\right)\right\}_0 s' = 0$$

or

$$\frac{d^2 s'}{dt^2} + p_1^2 s' = 0, \tag{21}$$

Where

$$p = \sqrt{\frac{\left(\dot{\phi}^2 r_s^3 - qGM_s\right) r_E e}{2a^2\left(1 - e^2\right)^2 r_s^3}} \sqrt{\frac{\sqrt{k_1}}{W_0 k_0}} \tag{22}$$

$$W_0 = (W)_0 = \left\{\frac{\partial}{\partial k}\left(n\frac{\partial u}{\partial s}\right)\frac{\partial u}{\partial s} - n\frac{\partial^2 u}{\partial s^2}\frac{\partial u}{\partial k}\right\}_0$$
$$= \left\{\sqrt{k_1}\operatorname{Cos}^2(n't + \varepsilon)\right\}_0 = \sqrt{k_1}\operatorname{Cos}^2\left(\dot{\phi} t + \varepsilon_0\right).$$

The solution of Eqn. (21) is given by

$$s' = A\operatorname{Sin}(p_1 t + \lambda_\circ) \tag{23}$$

Where $A = \sqrt{k_2}/p$, $k_2, \lambda_0 =$ Constants of integration, $s' = s - n't$. The Eqn. for s gives,

$$s = n't + A\operatorname{Sin}(p_1 t + \lambda_0) \tag{24}$$

$$k = k_0 + HA'(k/W)_0 \frac{A}{p_1}\operatorname{Cos}(p_1 t + \lambda_0) \tag{25}$$

Where k_0 is determined from $n_0 = n'$. Since n_0 is a known function of k_0. The amplitude 'A' and the time period T are given by,

$$A = \sqrt{k_2}/p_1, \quad T = 2\pi/p_1,$$

Where k_2 is an arbitrary constant,

$$p = \sqrt{\frac{\left(\dot{\phi}^2 r_s^3 - qGM_s\right) r_E e}{2a^2\left(1 - e^2\right)^2 r_s^3 k_0 \operatorname{Cos}\left(\dot{\phi} + \varepsilon_0\right)}}.$$

Using Eqn. (13), k_0 may be written as $k_0 = \sqrt{k_1}/n_0$. We may choose the constants of integration as $k_1 = 1$, $k_2 = 1$, $\varepsilon_\circ = 0$. The amplitude and time period are given by,

$$A = \frac{2\sqrt{2a^2\left(1 - e^2\right)^2 r_s^3}}{\sqrt{\left(\dot{\phi}^2 r_s^3 - qGM_s\right) r_E e n_0}} \operatorname{Cos}\phi,$$

$$T = \frac{4\pi\sqrt{2a^2\left(1 - e^2\right)^2 r_s^3}}{\sqrt{\left(\dot{\phi}^2 r_s^3 - qGM_s\right) r_E e n_0}} \operatorname{Cos}\phi.$$

In the same manner we have calculated amplitudes and time periods at other points. Thereafter two cases arise:

- Case 1: If we take only solar radiation pressure as perturbing forces, then there are only three points at which resonance occurs. Corresponding amplitudes and time-periods are given in Table 2 below.
- Case 2: In addition to the above, if we consider velocity dependent terms of P-R drag, then five points of resonance occur where four points of resonance are same as in case 1, and 1:2 and 3:2 resonances occur only due to velocity dependent terms of P-R drag. But amplitudes and time-periods at all resonance points are not same as in the case of solar radiation pressure. Corresponding amplitude and time-period are given in Table 3.

Table 2: Ai's and Ti's, at resonance points with only radiation pressure as perturbing force

Resonance	Amplitude	Time period
$n = \dot{\varphi}$	$A1, A2$	$T1, T2$
$2n = \dot{\varphi}$	$A5$	$T5$
$3n = \dot{\varphi}$	$A9$	$T9$

Table 3: Ai's and Ti's, at resonance points with velocity dependent terms of P - R drag as perturbing force

Resonance	Amplitude	Time period
$n = \dot{\varphi}$	$A3, A4$	$T3, T4$
$2n = \dot{\varphi}$	$A6$	$T6$
$n = 2\dot{\varphi}$	$A7, A8$	$T7, T8$
$3n = 2\dot{\varphi}$	$A10$	$T10$

Values of A_i's & T_i's are given in Appendices B & C.

4. Discussion and conclusion

We have investigated the resonance in the motion of a satellite in the Earth-Sun system due to P-R drag, by using a special method of Brown and shook (1933). After deducing Equation of motion of the geocentric satellite in vector as well as in polar form, it is found that there are five points $R_1\left(n=\dot{\phi}\right)$, $R_2\left(3n=\dot{\phi}\right)$, $R_3\left(2n=\dot{\phi}\right)$, $R_4\left(3n=2\dot{\phi}\right)$ and $R_5\left(n=2\dot{\phi}\right)$ at which resonances occurs, where $n\approx\dot{\theta}_0$ the angular velocity of a satellite is and $\dot{\varphi}$ is average angular velocity of the Earth. The 1:1 resonance occurs four times, 2:1, 1:2 resonance occur twice while 3:1 and 3:2 resonance occurs once only. There are two resonance points 3:2 and 1:2 occur only due to velocity dependent terms of P-R drag. If we ignore this perturbing force then resonance will occur only at three points in the equation of motion of a satellite. Using the satellite data, a = 6921000m; e = 0065; $\varphi^{\cdot}$ = 0.15695 °/s, r_s = 149599 × 10^6m, r = 149.6 × 10^9m, c = 3×10^8 m/s, we can make the quantities dimensionless by taking $M_S+M_S=1$, $G=1$, $r_S=1$.

From the expression of amplitude and A_1 and time period T_1 it is clear that A_1 and T_1 are periodic. From Fig. 2(a) and 2(b) we observe that amplitudes and time period increases when q increases and it is maximum at $\phi=0$. From Eqn. (3) $(1-q)$ is the factor of velocity dependent terms of P-R drag, when q increases $(1-q)$ decreases then effect of velocity dependent terms of P-R decreases and hence when P-R decreases then amplitude as well as time period increases. Figs. 3(a) and 3(b) also explain the amplitude and time period with respect to ϕ. In this case it can be observed that amplitude become very high of greater range of ϕ. But not in the case of velocity dependent terms of P-R drag. Similarly Fig. 3c explain the variation of amplitude A_4 for $0<\phi<90^0$ and $0<q<1$ at resonance 1:2. Fig. 3(c) shows that amplitude is periodic with respect to ϕ and it increases for an increase in q and vice-versa. When velocity dependent terms of P-R decreases then amplitude increases. The present study is becoming of more interest in the commensurability orbits, for example navigation satellite system.

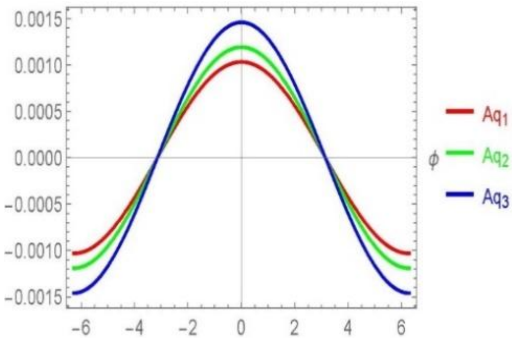

Fig. 2(a): Variation in amplitudes for 0°< ϕ<90° at q_1 = 0.25 (red), q_2 = 0.45 (green), q_3 = 0.65 (blue)

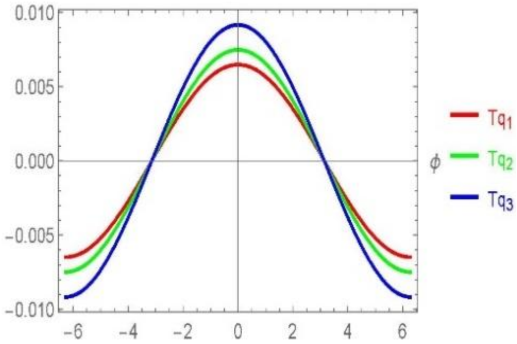

Fig. 2(b): Variation in time period, for 0°< ϕ<90° at q_1 = 0.25 (red), q_2 = 0.45 (green), q_3 = 0.65 (blue)

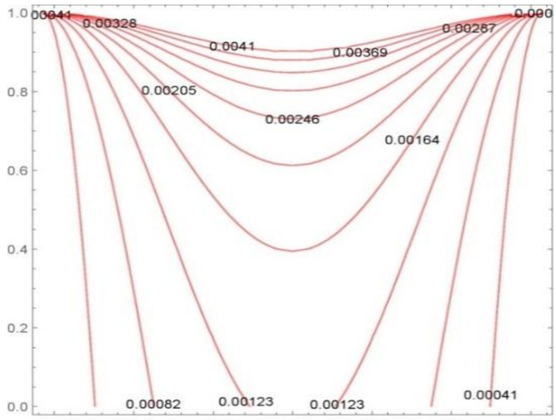

Fig. 3(a): Variation in amplitudes for - 1° < ϕ < 1° and 0 < q < 1 at resonance 1:1

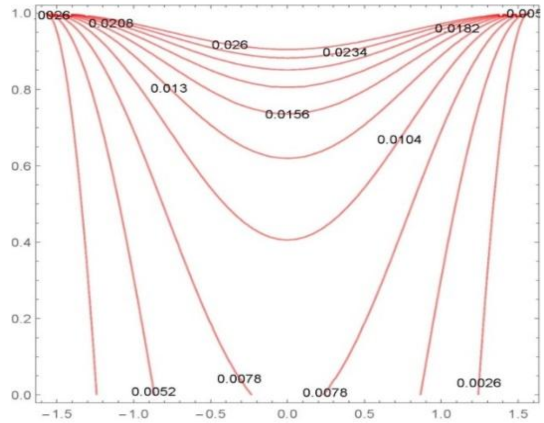

Fig. 3(b): Variation in time periods for - 1° < ϕ < 1° and 0 < q < 1 at resonance 1:1

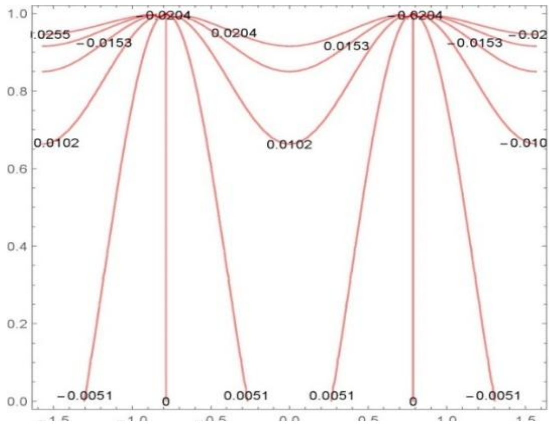

Fig. 3(c): Variation in amplitudes for - 90° < ϕ < 90° and 0 < q < 1 at resonance 1:2

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APPENDIX A:

$$C_1=-\frac{qGM_s}{r_s^3a\left(1-e^2\right)}+\frac{GM_E}{r_0^4},C_2=-\frac{F_g r_E^2\dot{\theta}_0 e\sin^2\alpha_0}{8cr_s a\left(1-e^2\right)},C_3=-\frac{F_g r_E^2\dot{\theta}_0\sin^2\alpha_0}{4cr_s a\left(1-e^2\right)},C_4=-\frac{F_g r_E^2\dot{\theta}_0 e\sin^2\alpha_0}{8cr_s a\left(1-e^2\right)},C_5=-\frac{er_E\left(\dot{\phi}^2r_s^3-qGM_s\right)}{r_s^3a^2\left(1-e^2\right)^2},$$

$$C_6=-\frac{\left(1-q\right)F_g\left(\dot{\phi}-\dot{\psi}_0\right)er_E}{ca^2\left(1-e^2\right)^2},C_7=-\frac{\left(1-\cos\alpha_0\right)r_E\left(\dot{\phi}^2r_s^3-qGM_s\right)}{2r_s^3a^2\left(1-e^2\right)^2},C_8=\frac{\left(1-q\right)\left(\cos\alpha_0-1\right)F_g r_E}{2c}\left\{\frac{\dot{\theta}_0+\phi-\psi}{r_s}-\frac{\dot{\phi}-\dot{\psi}_0}{a^2\left(1-e^2\right)^2}\right\},$$

$$c_9=\frac{-\left(1+\cos\alpha_0\right)r_E\left(\dot{\phi}^2r_s^3-qGM_s\right)}{2r_s^3a^2\left(1-e^2\right)^2}c_{10}=\frac{-\left(1-q\right)F_g\left(1+\cos\alpha_0\right)r_E}{2cr_s}\left\{\frac{\dot{\theta}_0-\dot{\phi}}{r_s}+\frac{\dot{\phi}}{ca^2\left(1-e^2\right)^2}\right\},c_{11}=\frac{-\left(1-\cos\alpha_0\right)r_E\left(\dot{\phi}^2r_s^3-qGM_s\right)}{2r_s^3a^2\left(1-e^2\right)^2},$$

$$c_{12}=-\frac{F_g\left(1-\cos\alpha_0\right)r_E\left(1-q\right)e\left(\dot{\phi}-\dot{\psi}_0\right)}{2ca^2\left(1-e^2\right)^2},c_{13}=-\frac{\left(1+\cos\alpha_0\right)r_E\left(\dot{\phi}^2r_s^3-qGM_s\right)}{2cr_s^3a^2\left(1-e^2\right)^2},c_{14}=-\frac{F_g\left(1+\cos\alpha_0\right)r_E\left(1-q\right)e\left(\dot{\phi}-\dot{\psi}_0\right)e}{2ca^2\left(1-e^2\right)^2},$$

$$c_{15}=-\frac{r_E^2\left(1-q\right)\left(\dot{\phi}-\dot{\psi}_0\right)\sin^2\alpha_0F_g}{4cr_s a\left(1-e^2\right)},c_{16}=\frac{-\left(1-q\right)F_g r_E^2\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)}{4cr_s a\left(1-e^2\right)}\left(\cos\alpha_0-0.5-0.5\cos^2\alpha_0\right),c_{17}=-\frac{\left(1-q\right)r_E^2F_g\left(\dot{\phi}-\dot{\psi}_0\right)}{8cr_s a\left(1-e^2\right)}$$

$$\left(\cos\alpha_0+0.5+0.5\cos^2\alpha_0\right),c_{18}=-\frac{\left(1-q\right)er_E^2F_g}{8cr_s a\left(1-e^2\right)}\left\{\dot{\theta}_0\left(0.5-\cos\alpha_0+0.5\cos^2\alpha_0\right)-\left(\dot{\phi}-\dot{\psi}_0\right)\left(\cos\alpha_0-1.5+0.5\cos^2\alpha_0\right)\right\},$$

$$c_{19}=-\frac{\left(1-q\right)er_E^2F_g}{16cr_s a\left(1-e^2\right)}\left\{\dot{\theta}_0\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)+\left(\dot{\phi}-\dot{\psi}_0\right)\left(2\cos\alpha_0+3-\cos^2\alpha_0\right)\right\},c_{20}=\frac{\left(1-q\right)er_E^2F_g}{8cr_s a\left(1-e^2\right)}\left(\dot{\theta}_0+\dot{\phi}-\dot{\psi}_0\right)$$

$$\left(\cos\alpha_0-0.5-0.5\cos^2\alpha_0\right),c_{21}=-\frac{\left(1-q\right)er_E^2F_g}{8cr_s a\left(1-e^2\right)}\left(\dot{\theta}_0-\dot{\phi}+\dot{\psi}_0\right)\left(\cos\alpha_0+0.5+0.5\cos^2\alpha_0\right),C_{22}=-\frac{qGM_se}{r_s^3a\left(1-e^2\right)}.$$

APPENDIX B:

$$A_1=\frac{\sqrt{2a^2\left(1-e^2\right)^2 r_s^3}}{\sqrt{\left(\dot{\phi}^2 r_s^3-qGM_s\right)r_E e n_0}}\cos\phi, A_2=\frac{2a\left(1-e^2\right)\sqrt{r_s^3}}{\sqrt{\left(\dot{\phi}^2 r_s^3-qGM_s\right)r_E\left(1+\cos\alpha_0\right)e n_0}}\cos\phi,$$

$$A_3=\frac{4\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 e(1-q)F_g\left(n_0\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)-\dot{\phi}\left(3-\cos^2\alpha_0+2\cos\alpha_0\right)\right)}}\cos\phi,$$

$$A_4=\frac{4\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0 e(1-q)F_g\left(n_0-\dot{\phi}\right)\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)}}\cos\phi, A_5=\frac{2a\left(1-e^2\right)\sqrt{r_s^3}}{\sqrt{r_E n_0\left(\dot{\phi}^2 r_s^3-qGM_s\right)\left(1+\cos\alpha_0\right)}}\cos\frac{\phi}{2},$$

$$A_6=\frac{4\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0 e(1-q)\left(n_0-\dot{\phi}\right)F_g\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)}}\cos\frac{\phi}{2}, A_7=\frac{2\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0(1-q)\left(+\dot{\phi}-\dot{\psi}\right)F_g\sin^2\alpha_0}}\cos 2\phi,$$

$$A_8=\frac{4\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0 e\dot{\phi}\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)}}\cos 2\phi, A_9=\frac{2a\left(1-e^2\right)\sqrt{r_s^3}}{\sqrt{r_E n_0 e\left(\dot{\phi}^2 r_s^3-qGM_s\right)\left(1+\cos\alpha_0\right)}}\cos\phi,$$

$$A_{10}=\frac{4\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0 e\dot{\phi}\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)}}\cos\frac{2\phi}{3},$$

APPENDIX C:

$$T_1=\frac{2\pi\sqrt{2a^2\left(1-e^2\right)^2 r_s^3}}{\sqrt{\left(\dot{\phi}^2 r_s^3-qGM_s\right)r_E e n_0}}\cos\phi, T_2=\frac{4\pi a\left(1-e^2\right)\sqrt{r_s^3}}{\sqrt{\left(\dot{\phi}^2 r_s^3-qGM_s\right)r_E\left(1+\cos\alpha_0\right)e n_0}}\cos\phi,$$

$$T_3=\frac{8\pi\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 e(1-q)F_g\left(n_0\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)-\dot{\phi}\left(3-\cos^2\alpha_0+2\cos\alpha_0\right)\right)}}\cos\phi,$$

$$T_4=\frac{8\pi\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0 e(1-q)F_g\left(n_0-\dot{\phi}\right)\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)}}\cos\phi, T_5=\frac{4\pi a\left(1-e^2\right)\sqrt{r_s^3}}{\sqrt{r_E n_0\left(\dot{\phi}^2 r_s^3-qGM_s\right)\left(1+\cos\alpha_0\right)}}\cos\frac{\phi}{2},$$

$$T_6=\frac{8\pi\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0 e(1-q)\left(n_0-\dot{\phi}\right)F_g\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)}}\cos\frac{\phi}{2}, T_7=\frac{4\pi\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0(1-q)\left(\dot{\phi}-\dot{\psi}\right)F_g\sin^2\alpha_0}}\cos 2\phi,$$

$$T_8=\frac{8\pi\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0 e\dot{\phi}\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)}}\cos 2\phi, T_9=\frac{4\pi a\left(1-e^2\right)\sqrt{r_s^3}}{\sqrt{r_E n_0 e\left(\dot{\phi}^2 r_s^3-qGM_s\right)\left(1+\cos\alpha_0\right)}}\cos\frac{\phi}{3},$$

$$T_{10}=\frac{8\pi\sqrt{2ac\left(1-e^2\right)r_s}}{\sqrt{r_E^2 n_0 e\dot{\phi}\left(1+\cos^2\alpha_0+2\cos\alpha_0\right)}}\cos\frac{2\phi}{3},$$