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A Numerical model for Stability Analysis of Pre-cracked Beam-columns

Peter N. Jiki[1*], Usman Karim[2]

[1]*Department of Civil Engineering University of Agriculture, PMB 2373, Makurdi, Benue State, Nigeria*
[2]*Department of Mathematics and Computer Science, Benue State University Makurdi, Benue State Nigeria*
*peterjiki@yahoo.com

Abstract

The proposed research paper reports results of the stability analysis of pre-cracked beam-columns. A stiffness reduction parameter due to pre-crack is first calculated, which is used in the equilibrium equations for buckling analysis. Stiffness and stability matrices are derived from the resulting equilibrium equations using the finite difference procedure. An object oriented code in java is developed based on the inverse power method for the extraction of the smallest eigen value corresponding to the critical load. The calculated parameter is then used to calculate the reduced buckling load due to pre-crack. Results obtained compare well with published results in the literature. It is concluded that the parameter k is a good indicator for monitoring stiffness degradation due to pre-crack and that java programming language which is mainly used for commercial and internet applications is a candidate tool for fracture mechanics computations.

Keywords**:** Pre-crack; Stability; Beam-columns; Java code; Stiffness reduction.

Introduction

It is well known that imperfections such as cracks and initial crocked nature reduce the load carrying capacity of columns and beam-columns (Liebowitz *et al.*, 1967; Liebowitz & Claus, 1968; Gounaris *et al.*, 1995; Bentham & Koiter, 1973; Okamura, *et al.*, 1969; Chondros & Dimarogonas, 1989; Knott & Elliot, 1979; Jiki, 2007). The presence of cracks in structures has the consequence of changing the dynamic characteristics such as changes in mass distribution, natural frequencies which can lead to resonance and damping properties. Modification of stress fields due to cracks and the decaying nature of these stress fields with distance away from the crack location, renders quantification of these defects for the purpose of failure analysis difficult (Liebowitz *et al.*, 1967; Liebowitz & Claus, 1968; Gounaris *et al.*, 1995; Bentham & Koiter, 1973; Okamura *et al.*, 1969; Chondros & Dimarogonas, 1989; Knott & Elliot, 1979; Jiki, 2007; Capuani & Willis 1997; Papadopolus, 1992; Anifantis & Dimarogonas, 1983). Cracks also introduce a zone of discontinuity in the region of interest, which makes choice of solutions and analytical modeling of the problem difficult.

Analytical researches on buckling of circular rings and columns with cracks have been reported by Dimarogonas (Dimarogonas, 1981), using perturbation method (Liebowitz *et al*., 1967; Liebowitz & Claus, 1968), used sine and cosine functions to model buckling loads for pre-cracked columns. Their model was approximate as it disregarded the discontinuity of the sine and cosine functions in their study. Capuani & Willis (1997) have used asymptotic expansions to study wave propagation behaviour of cracked structures and concluded that the presence of a crack introduces wave scattering at the crack location. Analytical studies on conservatively loaded beam-columns have been reported by Jiki (2007) who used Liapunov's second method of stability analysis in conjunction with eigenvalue in equalities to obtain reduced load carrying capacities of pre-cracked beam-columns and concludes that the presence of cracks accelerate the process of failure either by buckling or by fracture.

Numerical studies employing the finite element method to study pre-cracked structures have been reported by: (Papadopolus, 1992; Chondros & Dimarogonas, 1989). It is interesting to point out here that most of the finite element programs used to process the above cited numerical studies used procedural codes mainly using fortran. However, for the past decade or so, attempt has been made by shifting emphasis on procedural codes to object-oriented codes to process the finite element equations as can be seen from works by (Mackie, 2001; Jiki, 2009; Nikishkov, 2006).

However, to the best knowledge of the present writers, no numerical code using Java programming language has been used to process the stability characteristics such as eigenvalues and eigen vectors for pre-cracked columns or beam-columns employing either the finite element, boundary element or the finite difference methods.

The choice of Java programming language as a tool for processing our eigenvalue equations is in line with current development in computational technology where object-oriented programming is considered more efficient in terms of program modification, code reuse, data abstraction, encapsulation and program extendibility. More so, a java code is platform independent as the compiler comes with a complete Java Development Kit (JDK). Thus, the purpose of the present work is to use the newly developed java code to process the stability characteristics of the pre-cracked columns of interest.

Contribution:

(a) Stiffness reduction parameter k due to crack is proposed.
(b) A new object oriented code in java for stability analysis is announced in Jiki (2007).
(c) A specific class coded in java for rapid calculation of the proposed stiffness reduction parameter k is proposed to extend the code reported in Jiki (2007) and is shown here in appendix A.

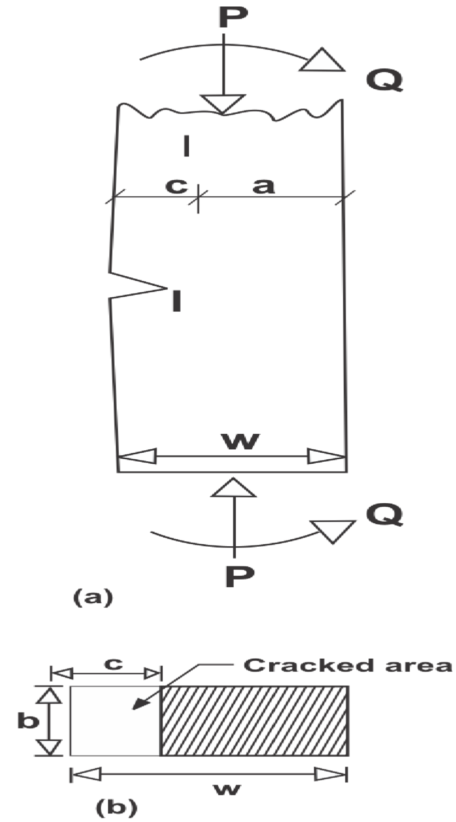

Fig. 1. Pre-Cracked Beam-Column Model: (a) Edge crack in a rectangular strip loaded by axial load P and applied Moment Q: (b) Cracked Section of the Stip.

Calculation of stiffness reduction parameter k

Consider an open thin-walled column with section as shown in fig.1. The column has width w, length L and thickness b. For such a structural member with an edge crack Erwin and Griffith (Knott & Elliot, 1979) have shown that the compliance derivative is related to the strain energy release rate of the cracked body in bending as:

$$G_1 = \frac{1}{2} Q^2 \frac{dF}{dA} \qquad (1)$$

In which

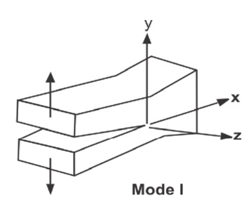

Fig. 2. Failure mode for calculation of stress intensity factor k_1

Q provides the bending tension opening up the crack.

F is local compliance (flexibility) due to crack

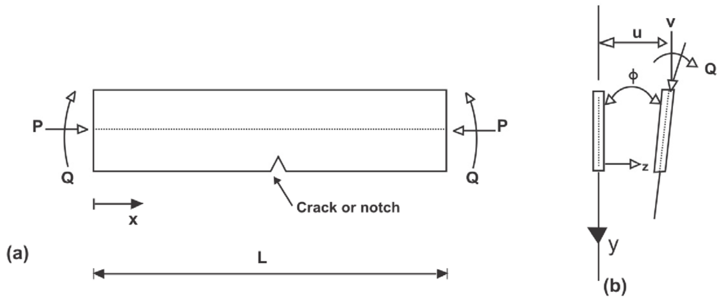

Fig. 3. Thin bar under axial load P and uniform bending moment Q: (a) Bean-column with applied loads; (b) Section of the deflected column

A is surface area of the cracked section.

The stress intensity factor k_1 for mode 1 opening (failure) as shown in fig.2 for the cracked body is defined by Knott & Elliot (1979) as a measure of the magnitude of crack tip singularity. It is related to the strain energy release rate G_1 of the cracked column as:

$$G_1 = \frac{1-\upsilon^2}{E} k_1^{\ 2} (\forall\ plain\,strain) \qquad (2)$$

in which

υ is Poisson's ratio

E is Young's modulus

From equations (1) and (2) we have

$$\frac{1}{2} Q^2 \frac{dF}{dA} = \frac{1-\upsilon^2}{E} k_1^{\ 2} (\forall\ plain\,strain) \qquad (3)$$

It is assumed in the present work that the column buckles due to (Pδ) moment Q. Q can also be provided by eccentricity of loading. Q is needed to provide the driving force to open up the crack leading to crack propagation.

The mode 1 opening stress intensity k_1, is given by (Bentham & Koiter, 1973) as:

$$k_1 = \frac{6Q}{a} k_0 (\frac{c}{aw})^{1/2} \qquad (4)$$

In which k_o is dimensionless stress intensity factor given as [9]

$$k_0 = 1.122[1 - 2.217(\frac{c}{w}) + 0.523(\frac{c}{w})^2] \qquad (\forall (\frac{c}{w}) \to 0) \qquad (5)$$

H is the width of the strip or column section.

a = w – c (see fig.1)

c = crack length

Q = moment opening crack.

A buckling problem is a plain strain problem and by using equation (3), the compliance derivative of the cracked section is given as:

$$\frac{dF}{dA} = \frac{2(1-\upsilon^2)}{EQ^2} k_1^{\ 2} \qquad (6)$$

The surface area of the cracked section per unit width of crack is given as:

$$A = c\,x1 \qquad (7)$$

and

$$\frac{dA}{dc} = 1 \qquad (8)$$

Therefore

$$\frac{dF}{dc} = \frac{2(1-\upsilon^2)}{EQ^2} k_1^{\ 2} \qquad (9)$$

Usually it is expected that for buckling, $k_1 << k_{1b}$ in which k_{1b} is the buckling stress intensity factor.

From equation (4) we have:

$$k_1^{\ 2} = \frac{36 Q^2 k_1^{\ 2}}{a^3} (\frac{c}{w}) \qquad (10)$$

Substitution of equation (10) into equation (9) and integrating gives the compliance as:

$$F = \frac{72(1-\upsilon^2)}{Ea^3} \int k_0^{\ 2} (\frac{c}{w}) dc \qquad (11)$$

in which

$$k_0^{\ 2} = \{1.122[1 - 2.217(\frac{c}{w}) + 0.523(\frac{c}{w})^2]\}^2 \qquad (12)$$

Therefore

$$\int_0^c k_0^{\ 2} (\frac{c}{w}) dc = c[-1.25888 + 4.098(\frac{c}{w}) - 2.6339(\frac{c}{w})^2$$
$$+ 1.15968(\frac{c}{w})^3 - 0.7845(\frac{c}{w})^4 + 0.579(\frac{c}{w})^5] \qquad (13)$$

Substitution of equation (13) into equation (11) gives the local compliance F as:

$$F = \frac{72(1-\nu^2)c\xi}{Ea^3} \qquad (14)$$

In which

$$\xi = [-1.25884 + 4.0498(\frac{c}{w}) - 2.6339(\frac{c}{w})^2$$
$$+1.15968(\frac{c}{w})^3 - 0.7845(\frac{c}{w})^4 + 0.579(\frac{c}{w})^5 \quad (15)$$

If F is the compliance of a cracked body obtained by stress function method and F_0 is the compliance of the same body obtained by crack mouth opening displacement (COD) then the non-dimensional compliance

$$F^* = \frac{F}{F_0} = \frac{1}{k} \quad (16)$$

Now

$$F_0 = \frac{v}{Q}$$
$$= \frac{\textit{crack mouth opening displacement}}{\textit{Applied moment}} \quad (17)$$

Remark 1. The assumption made for this case of edge crack is that failure will be due to buckling by crack mouth opening at one side providing maximum opening on that side during bending of the column. To achieve this (Liebowitz & Claus, 1968), have to be used load eccentricity. In this work we use either load eccentricity or $p\delta$ to achieve the above goal.

Thus for the edge crack shown in fig. 3a, it can be shown that (Liebowitz & Claus, 1968).

$$2v = \frac{4\sigma}{E}(1-\upsilon^2)(c^2 - x^2)^{1/2} \quad (18)$$

For maximum v we have $x = 0$. Equation (18) becomes:

$$v = \frac{2\sigma c}{E}(1-\upsilon^2) \quad (19)$$

Substitution of equation (19) into equation (17) we have

$$\frac{v}{Q} = \frac{2\sigma c}{EQ}(1-\upsilon^2) \quad (20)$$

but

$$\sigma = \frac{Q}{Z} \quad (21)$$

Therefore $\frac{v}{Q} = \frac{2c(1-\upsilon^2)}{EZ} = F_0 \quad (22)$

In which Z is the elastic modulus of the section given for a rectangular section as:

$$Z = \frac{bw^2}{6}(\textit{the strip shown in figure } 1) \quad (23)$$

Thus the non dimensional compliance

$$F^* = \frac{1}{k} = \frac{72(1-\upsilon^2)c\xi}{Ea^3} x \frac{EZ}{2c(1-\upsilon^2)} \quad (24)$$

$$F^* = \frac{36Z\xi}{a^3} \quad (25)$$

Substitution for Z from equation (23) into equation (25) gives:

$$F^* = \frac{6bw^2\xi}{a^3} \quad (26)$$

But from fig. 3b and equation (7) we have:

$$a^3 = (w^3 - 3w^2c + 3wc^2 - c^3) \quad (27)$$

Therefore the non dimensional stiffness

$$k = \frac{(w^3 - 3w^2c + 3wc^2 - c^3)}{6b^2w^2\xi} \quad (28)$$

By dividing equation (28) top and bottom by w^3 gives:

$$k = \frac{[1 - 3(\frac{c}{w}) + 3(\frac{c}{w})^2 - (\frac{c}{w})^3]}{6(\frac{b}{w})\xi} = \frac{Y_1}{Y_{11}} \quad (29)$$

in which

$$Y_1 = [1 - 3(\frac{c}{w}) + 3(\frac{c}{w})^2 - (\frac{c}{w})^3] \quad (30)$$

$$Y_{11} = 6(\frac{b}{w})\xi \quad (31)$$

Finance Difference Model

The buckling loads for axially loaded non uniform columns which buckle in flexure can be obtained from the governing differential equation for small deflection theory of bending as (Iremonger, 1980):

$$EI_x \frac{d^2y}{dx^2} = -M_x \quad (32)$$

In which

E is Young's modulus, x is distance along the length of the column measured from one end and I_x is the second moment of area about the mid axis of the non uniform column and M_x is bending moment. When our proposed stiffness reduction parameter k of equation (29) is introduced into equation (32) we have:

$$EI_x(1-k)\frac{d^2y}{dx^2} = M_x \quad (33)$$

Then the finite difference form of equation (33) is written as [15]:

$$EI_x(1-k)(y_{i-1} - 2y_i + y_{i+1}) = M_x \qquad (34)$$

In which y_{i-1}, y_i and y_{i+1} are lateral displacements at three stations covering two segments length h for a column of length L which is divided into L/h equal segments. Application of equation (34) and other similar difference equations at a number of stations yields a set of homogeneous equations leading to the matrix eigen-value equation of the form

$$[A]\{y\} = \lambda[B]\{y\} \quad (35)$$

In which

[A] and [B] are matrices, {y} is eigen vector and λ is an eigen value. Solution of equation (35) by the inverse power method of matrix iteration leads to the smallest eigen value corresponding to the lowest (critical) load for the column. To show the form of finite difference equations leading to matrices [A] and [B] we consider a fixed-free column from ref (Iremonger, 1980) as shown in figure 4 of the present work. The bending equilibrium equation for this column is (Iremonger, 1980):

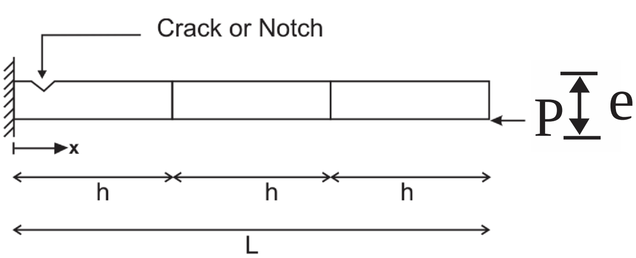

Fig. 4. Finite difference descretization of a cantilever column with load P at eccentricity e and and edge crack

$$\lambda = \frac{Pl^2}{\pi^2 EI(1-k)} \qquad (39)$$

$$EI_x(1-k)\frac{d^2y}{dx^2} = M_x = P(y_o - y) \quad (36)$$

In finite difference form equation (35) is written as:

$$y_{i-1} - 2y_i + y_{i+1} + \lambda(y_o - y_i) = 0 \quad (37)$$

In which

$$\lambda = \frac{Pl^2}{\pi^2 EI} = \frac{P}{P_e} \quad (\forall\, l = \frac{L}{3}, \textit{and no crack}) \quad (38)$$

By using our derived stiffness reduction parameter k into equation (38) we have:

The boundary conditions for the fixed –free and other column types are available in ref (Iremonger, 1980) and are not reproduced here. Equation (37) is further discredited as follows:

$$\left.\begin{aligned} y_0 - 2y_1 + y_2 - \lambda(y_0 - y_1) = 0 \\ 2y_1 - 2y_2 + y_3 - \lambda(y_0 - y_2) = 0 \\ 3y_2 - 2y_3 + y_4 - \lambda(y_0 - y_3) = 0 \end{aligned}\right\} (40)$$

Equation (40) is written in matrix notation as (Iremonger, 1980):

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} y_0 \\ y_1 \\ y_2 \end{Bmatrix} = \lambda \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} y_0 \\ y_1 \\ y_2 \end{Bmatrix} \quad (41)$$

Equation (41) is exactly in the form of equation (35). This completes our finite difference discretization of the cracked non uniform column. Equation (41) is transformed to standard eigen value format for solution by the inverse power method.

Characteristics of pre-cracked columns

In this section we study the effect of pre-crack on strength and deflections of columns. We use

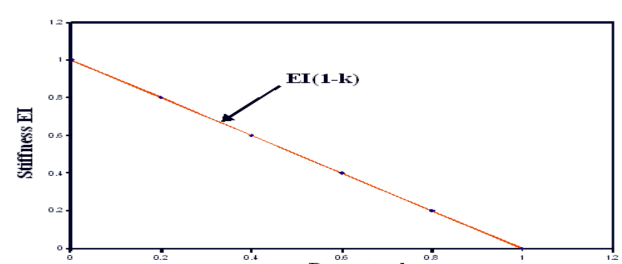

Fig. 5. Linear Relationship for stiffness decay in a cracked column

ideas presented in the previous sections to achieve our goal. In section 2 of the present work we have derived a parameter k that reduces the stiffness of a cracked column progressively as the crack grows or propagates to failure. Figure 5 shows plot of relationships between the stiffness reduction parameter k and the flexural stiffness EI which in this case is a linear relationship.

One of the effects of an edge crack on the strength of a beam column in flexure is that it reduces the flexural stiffness of the cracked beam column making it more flexible to deflect under the applied load. In order to study this particular effect (Okamula *et al.*, 1969), have proposed an equation for the calculation of deflection in pre-cracked columns and beam columns under eccentric axial load as:

$$\frac{\delta}{e}=\frac{1}{\cos\alpha L-\beta\alpha\sin\alpha L}-1 \quad (42)$$

or

$$\frac{\delta}{e}+1=\frac{1}{\cos\alpha L-\beta\alpha\sin\alpha L} \quad (43)$$

In which

$$\alpha=\sqrt{\frac{P}{EI}} \quad (44)$$

β is crack parameter, δ is deflection, e is eccentricity needed to open the crack, L is effective length of the column. However, in order to calculate the deflection of pre-cracked beam columns and columns using our proposed crack parameter k, we modify equation (42) by Okamula *et al.* (1967) as follows:

Let

$$\beta\alpha L\tan\alpha L=k \quad (45)$$

in which

k is our proposed mode 1 crack opening parameter. Then we write equation (42) in our method as:

$$\frac{\delta}{e}+1=\frac{1}{\cos\alpha L(1-k)} \quad (46)$$

Equation (42) by (Okamula *et al.*, 1967) and our modified equation (46) are plotted in figure 6 and the comparison is very good. Still on the calculation of deflection, we consider a column

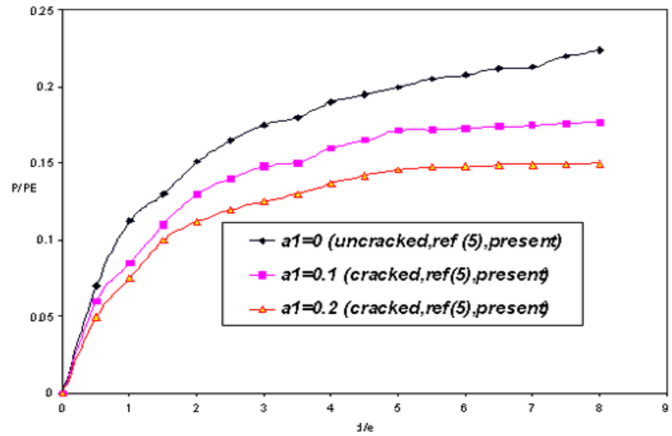

Fig.6. The deflection of a cracked column subjected to an eccentric compression load

which is simply supported at both ends with an initial imperfection of the form (Knowles, 1987):

$$y_0=a_0\sin\frac{\pi x}{L} \quad (47)$$

When the strut is loaded by a force P, the compressive stress at any cross section is given as the sum of $\frac{P}{A}$ *and* $\frac{M}{Z}$. The additional deflection at a point x from the origin is v and we have the equation of equilibrium for the uncracked column as [10]:

$$EI\frac{d^2y}{dx^2}=P(y+y_0)=M_x \quad (48)$$

or

$$\frac{d^2y}{dx^2}\alpha^2 y+\alpha^2 y_0\sin\frac{\pi x}{L}=0 \quad (49)$$

when the column has a crack at the middle in addition to an initial imperfection of amplitude a_0 we modify equation (48) as:

$$EI(1-k)\frac{d^2y}{dx^2}=M_x \quad (50)$$

Similarly equation (49) is modified to:

$$\frac{d^2y}{dx^2}\alpha_c{}^2 y^2+\alpha_c{}^2 a_0\sin\frac{\pi x}{L}=0 \quad (51)$$

in which

$$\alpha_c=\sqrt{\frac{P}{EI(1-k)}} \quad (52)$$

We observe here that equation (52) accounts for the presence of an edge crack in the column. If the total deflection in equation (48) is:

$$y + y_0 = d \qquad (53)$$

It can be shown that the amplification of the deflection in the un-cracked column is given as (Knowles, 1987) :

$$d = \frac{1}{(1 - \frac{P}{P_e})} a_0 \sin \frac{\pi x}{L} \qquad (54)$$

in which the amplification factor is $\frac{1}{(1 - \frac{P}{P_e})}$. The effect of crack is to amplify the deflection further to:

$$d = \frac{1}{(1 - \frac{P}{P_e})(1 - k)} a_0 \sin \frac{\pi x}{L} \qquad (55)$$

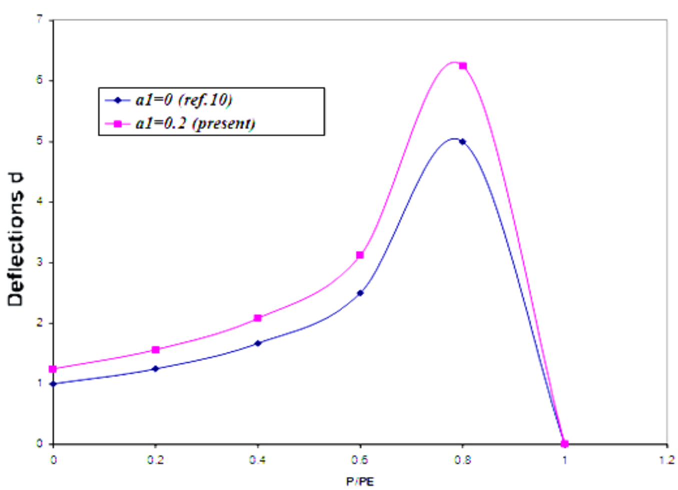

Fig. 7. Effect of pre-crack on deflection of beam- column.

Figure 7 shows plots of deflection against $\frac{P}{P_e}$ which further highlights additional amplification due to pre-crack for different values of parameter k.

Next we study the response characteristics of pre-cracked uniformly tapered columns. We use our proposed java code reported in Jiki (2009) to perform a convergence study which compares well with results of Iremonger (1980), when the column has no crack. Cases of pre-cracked columns are considered for values of the crack parameter k ranging from 0.0, 0.1, and 0.15 which are plotted in figure 7. As it is expected the presence of a crack re

Results

The main findings of our present study are as follows:

1. The presence of an edge crack at the middle of a column amplifies the deflection of the buckled column.
2. The presence of an edge crack at the middle of the column also accelerates the rate of buckling, that is, it leads to early buckling of the column.
3. The use of a derived stiffness reduction parameter k aids in determining the rate of decay of the pre-cracked stiffness of the column. The parameter can thus be used as a failure indicator.
4. A java class has been developed in this work for the purpose of calculating rapidly the proposed stiffness reduction parameter k. The class is now used to extend the java code reported in Jiki (2009) which has been used for stability calculations reported in the present work.
5. We have also found in the present study that the stiffness reduction parameter k derived in the present work is a good candidate for structural health monitoring applications. A java class called kclass for rapid calculation of the proposed parameter k is attached here in appendix A.

Conclusion

From the findings of the present work as detailed in section 7 we conclude as follows:

* A model for numerical stability analysis using the finite difference procedure is presented.

* A stiffness reduction parameter k is proposed and is used to calculate reduced buckling loads due to the presence of an edge crack in a pre-cracked beam-column.

* A linear relationship between reduced stiffness and the proposed parameter k has also been established.

* The derived parameter k presented here is a good candidate for future structural health monitoring applications.

* The present work also encourages the use of object oriented java codes for fracture mechanics application.

Appendix A

A class for calculating parameter k is separately attached here for clarity

```
import java.math.*;
import java.io.BufferedReader;
import java.io.IOException;
import java.io.InputStreamReader;
import java.text.DecimalFormat;
public class Kclass {
    /** Creates a new instance of Kclass */
   public Kclass() {
   }
   public static double calculatek (double b, double c, double w)
   {
      double y1 = (1 - 3*(c/w) + 3*Math.pow((c/w),2) - Math.pow((c/w),3));
      double y2 = (6*(b/w))*(-1.25884 + 4.0498 * (c/w) - 2.6339 * Math.pow((c/w),2) + 1.15968 * Math.pow((c/w),3) - 0.7845 * Math.pow((c/w),4) + 0.579 * Math.pow((c/w),5));
      double y3 = y1/y2;
      return y3;
   }
   public static void main (String [] args)
   {
      InputStreamReader sr = new InputStreamReader(System.in);
      BufferedReader br = new BufferedReader(sr);
      double b = 0.0, c = 0.0, w = 0.0;
      String txt = null;
      try
      {
         System.out.println("Enter the value of b");
         txt = br.readLine();
         b = new Double(txt).doubleValue();
         System.out.println("Enter the value of c");
         txt = br.readLine();
         c = new Double(txt).doubleValue();
         System.out.println("Enter the value of w");
         txt = br.readLine();
         w = new Double(txt).doubleValue();
      }
      catch (IOException ee){}
      System.out.println (" chk this "+ calculatek (b,c,w));
   }
}
```

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