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A Relation between Queue-Length Distributions during Server Vacations in Queues with Batch Arrivals, Batch Services, or Multiclass Arrivals: An Extension of Burke's Theorem

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Abstract

This paper presents a relation between queue-length distributions during vacations and those at vacation start and end times, in M/G/1 vacation queues with batch arrivals, batch services or multi-class arrivals. In order to obtain the relation, we develop a new approach that extends Burke's theorem to queues with batch arrivals, batch services or multi-class services. We also give an example to demonstrate how to apply the proposed relation to the analysis of such a queue.

Keywords: Batch Arrivals and Services, Burke's Theorem, Multi-class Arrivals, Queue-length Distributions, Vacation Queues

1. Introduction

It is well known that in the M/G/1 queue with server vacations and the exhaustive service policy, the following relation holds between the queue-length distribution during server vacations and that at epochs just after succesive vacations end:

$$P^V(z) = \frac{1-\Pi^T(z)}{E[N^T](1-z)} \tag{1}$$

where N^V is the number of customers during vacations and N^T is the number of customers at epochs just after successive vacation terminations. Also, the corresponding PGFs are defined as:

$$P^V(z) = E\left[z^{N^V}\right] \text{ and } \Pi^T(z) = E\left[z^{N^T}\right].$$

Similarly, if N^S denotes the number of customers at epochs just before successive vacations start and the corresponding PGF is defined as

$$\Pi^S(z) = E\left[z^{N^S}\right],$$

then the following relation holds in the M/G/1 vacation queue with non-exhaustive service policy.

$$P^V(z) = \frac{\Pi^S(z)-\Pi^T(z)}{E[N^T - N^S](1-z)}. \tag{2}$$

See[1] and [2] for the derivation and intuitive interpretation of 1 and 2.

Using 1 and 2, the queue-length distribution during vacations can be derived for a variety of M/G/1 vacation queues and the queue-length distribution at any arbitrary time, in turn, can be obtained from the well-known stochastic decomposition property:

$$P(z) = P^V(z)\cdot P(z)_{M/G/1}, \tag{3}$$

where $P(z)$ is the PGF of the queue length in the M/G/1 vacation queue in question and $P(z)_{M/G/1}$ is the PGF of the queue length in the standard M/G/1 queue[1,2].

It is natural to ask whether or not there exist such relations as in 1 and 2in the cases of M/G/1 vacation

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queues with batch arrivals, batch service or multi-class arrivals. This study explores the possibility of such relations in those queues. Using a generalization of Burke's theorem to those queues, this study demonstrates that there exist relations similar to 1 and 2 for M/G/1 vacation queues with batch arrivals, batch services or multi-class customers.

There have been recent studies on queue-length distributions in server vacation queues[10–15]. However, most works have only considered the queue-length distribution of a specific vacation queue as a special case. In this paper, we explore a general relation between queue-length distributions during vacations. Hence, this study is more related to the classic work on relations between queue-length distributions in vacation queues, such as[1–4]. However, those studies only considered relations between queue-length distributions in the standard M/G/1 vacation queue or M/G/1 vacation queues with batch arrivals or batch services separately. In this study, in a unified approach, we explore a general relation between queue-length distributions in various M/G/1 vacation queues with batch arrivals, batch services or multi-class arrivals.

In Section 2, we present a new way of extending Burke's theorem. In Section 3, we present our main result for the relation between queue-length distributions during vacations. In Section 4, we also give an example in order to demonstrate how to apply the proposed relation to the analysis of such a queue.

2. An Extention of Burke's Result

Consider G/G/c queues. Let $a(n)$ and $d(n)$, $n = 0,1,2,\ldots$, denote the probability that there are n customers just before successive customer arrivals, and the probability that there are n customers just after successive customer departures, respectively. Then, Burke's theorem says that

$$a(n) = d(n). \tag{4}$$

Let A denote a set of states in which the number of customers is less than or equal to n and A^C is the complementary set of A, i.e., a set of states in which the number of customers is greater than n. Then, Burke's theorem can be derived from the argument that, in a steady state, the transition rate from A to A^C needs to be identical to that from A^C to A. In other words, if $\gamma_{A,B}$ denotes the transition rate from set A to set B, then any stochastic model holds the following equation:

$$\gamma_{A,A^C} = \gamma_{A^C,A} \tag{5}$$

In a G/G/c queue, only one customer arrives at or departs from the system at a time. Thus, we have $\gamma_{A,A^C} = \lambda a(n)$ and $\gamma_{A^C,A} = \lambda d(n)$, where λ is the arrival rate of customers. From 5, we obtain Burke's theorem in 4. For the sake of simplicity, we will present Burke's theorem and its new variants in PGF form. Burke's theorem in 4 can be rewritten in PGF form as

$$\Pi^A(z) = \Pi^D(z), \tag{6}$$

where $\Pi^A(z) = \sum_{n=0}^{\infty} a(n)z^n$ and $\Pi^D(z) = \sum_{n=0}^{\infty} d(n)z^n$.

This standard form of Burke's theorem can be extended to queueing systems with batch arrivals or batch services or to those with both. $g(n)$ and $f(n)$, $n = 0,1,2,\ldots$, denote the probability that the size of a batch arrival is n and the probability that the size of a batch service is n, respectively. We assume that a member in batch arrival observes the customers in the system and those who are ahead of him/her in the batch. Similarly, a member in batch service observes the customers in the system and those who are behind him/her in the batch. Then, the standard argument for 6 can be generalized to the batch arrival and service case, and we have

$$\Pi^A(z)\frac{1-G(z)}{g(1-z)} = \Pi^D(z)\frac{1-F(z)}{f(1-z)} \tag{7}$$

where $G(z) = \sum_{n=0}^{\infty} g(n)z^n$, $g = \sum_{n=0}^{\infty} g(n)n$, $F(z) = \sum_{n=0}^{\infty} f(n)z^n$, and $f = \sum_{n=0}^{\infty} f(n)n$ (see[4].)

All these arguments for Burke's theorem so far are based on local balance equations that equate the level crossing rate from states that are less than or equal to n with that from states that are greater than n. However, if we employ an argument based on global balance equations that equate the transition rate from inside n with that from outside n, an extension of Burke's theorem in 7 can be obtained more straightforwardly. As we will demonstrate, an extension of Burke's theorem to multi-class queueing systems can also be easily obtained.

In order to compare the local balance approach and the global balance approach, we first derive 7 using our new

global balance approach. Let B denote a set consisting of only state n. Let λ_g be the rate of batch arrivals and λ_f the rate of batch services. A transition from B to B^C occurs when a batch arrival observes n customers in the system just before its batch arrival or a batch service with size j observes $n-j$ customers in the system just after its batch service. Thus, we have

$$\gamma_{B,B^C} = \lambda_g a(n) + \lambda_f \sum_{j=1}^{n} f(j) d(n-j). \quad (8)$$

Similarly, a transition from B^C to B occurs when a batch arrival with j observes $n-j$ customers in the system just before its batch arrival or a batch service observes n customers in the system just after its batch service. Thus, we have

$$\gamma_{B^C,B} = \lambda_g \sum_{j=1}^{n} g(j) a(n-j) + \lambda_f d(n). \quad (9)$$

Equating γ_{B,B^C} with $\gamma_{B^C,B}$, we have from 8 and 9

$$\lambda_g \Pi^A(z) + \lambda_f \Pi^D(z) F(z) = \lambda_g \Pi^A(z) G(z) + \lambda_f \Pi^D(z) . \quad (10)$$

Because in a steady state the average rate of customer arrivals is identical to the average rate of customer departures, we also have

$$\lambda_g g = \lambda_f f . \quad (11)$$

Substituting 11 into 10 and rearranging the terms in 10 gives 7.

We now extend our new global balance approach to multi-class queueing systems. We assume that there are P, $P \geq 1$, classes of customers. Customers arrives at the system in a batch, which can include different classes of customers. The customer distributions in batches are i.i.d. and independent of the batch arrival process. Customers are served one at a time, under an arbitrary service policy, so any sort of priority discipline can be assumed. X_i, $1 \leq i \leq P$, denotes the number of class-i customers in the batch; the corresponding PGF of the customer distributions in the batch is defined as

$$X(\mathbf{z}) \equiv E\left[(z_1)^{X_1} \cdots (z_P)^{X_P}\right].$$

The system state $\mathbf{n}$ is represented by a state vector $(n_1, \ldots, n_P)$, where n_i is the number of class-i customers in the system. $a(\mathbf{n})$ is the probability that a multi-class arrival batch observes the system state $\mathbf{n}$ just before its arrival, and $d_i(\mathbf{n})$ is the probability that a class-i customer observes the system state $\mathbf{n}$ just after its departure from the system.

Let B denote a set consisting of only state $\mathbf{n}$. A transition from B to B^C occurs when a multi-class batch observes state $\mathbf{n}$ just before its arrival, and when a class-i customer observes state $\mathbf{n}-\mathbf{e}_i$ just after its departure, where $\mathbf{e}_i$ is a P dimensional vector whose i-th element is 1 and all of whose other elements are 0. Hence, we obtain

$$\gamma_{B,B^C} = \lambda_X a(\mathbf{n}) + \sum_{j=1}^{P} \lambda_j d_j(\mathbf{n}-\mathbf{e}_j) , \quad (12)$$

where λ_X is the arrival rate of batches, and λ_j is the departure rate of class-j customers. Similarly, a transition rate from B^C to B occurs when a multi-class batch with $\mathbf{x} = (x_1, \ldots, x_P)$ customers observes state $\mathbf{n}-\mathbf{x}$ just before its arrival and when a class-i customer observes state $\mathbf{n}$ just after its departure. Hence we obtain

$$\gamma_{B^C,B} = \lambda_x \sum_{\mathbf{n}=\mathbf{l}+\mathbf{x}} g(\mathbf{x}) a(\mathbf{l}) + \sum_{j=1}^{P} \lambda_j d_j(\mathbf{n}) \quad (13)$$

Equating γ_{B,B^C} with $\gamma_{B^C,B}$, we have from 12 and 13

$$\lambda_X \Pi^A(\mathbf{z}) + \sum_{j=1}^{P} \lambda_j \Pi_j^D(\mathbf{z}) z_j = \lambda_X \Pi^A(\mathbf{z}) X(\mathbf{z}) + \sum_{j=1}^{P} \lambda_j \Pi_j^D(\mathbf{z}) . \quad (14)$$

where $\Pi^A(\mathbf{z}) = \sum_{n_1, \cdots n_P \geq 0} a(\mathbf{n}) (z_1)^{n_1} \cdots (z_P)^{n_P}$ and $\Pi_j^D(\mathbf{z}) = \sum_{n_1, \cdots n_P \geq 0} d_j(\mathbf{n}) (z_1)^{n_1} \cdots (z_P)^{n_P}$. Because in a steady state the average rate of class-j customer arrivals is identical to the average rate of class-j customer departures, we also have

$$\lambda_X E[X_j] = \lambda_j . \quad (15)$$

Substituting 15 into 14 and rearranging the terms gives a variant of Burke's theorem for multi-class G^X/G/c queues as follows:

$$\Pi^A(\mathbf{z})(1 - X(\mathbf{z})) = \sum_{j=1}^{P} \Pi_j^D(\mathbf{z})(1 - z_j) E[X_j] \quad (16)$$

If the customer distribution at epochs just after successive departures is given, we can easily obtain the customer distribution at epochs just before successive arrivals from 16. Also, if the arrival process is a Poisson process, we can also derive the customer distribution at any arbitrary time from the PASTA property[5].

In the next section, using the new, extended approach of Burke's theorem in this section, we derive relations between queue-length distributions during vacations and

those at epochs just before vacations start and end for $M^X/G/1$ vacation queues and multi-class M/G/1 vacation queues.

3. Queue-length Distributions During Vacations

Consider an M/G/1 queue with server vacations. The server vacation policy can be exhaustive or non-exhaustive. Let N denote the number of customers in the system. Let Y denote the server state: $Y=0$ when the server is on vacation and $Y=1$ when the server is busy. Then, the system state can be represented by (N,Y). We define a set C of states as $C=\{(n,0)\}$. Then, a transition from C to C^C occurs when a vacation ends with n customers in the system or a customer arrives at the system with n customers while the server is on vacation. Let $a^V(n)$ denote the queue-length distribution while the server is on vacation, $k(n)$ denote the queue-length distribution when a vacation ends, λ_V be the rate at which a customer arrives at the system during a vacation and β be the rate at which vacations start (terminate). Then, the global balance argument in Section 2 gives the following equation:

$$\gamma_{C,C^C}=\lambda_V a^V(n)+\beta k(n). \tag{17}$$

Similarly, a transition from C^C to C occurs when a vacation starts with n customers in the system or a customer arrives at the system with $n-1$ customers while the server is on vacation. Thus, if we let $h(n)$ denote the queue-length distribution when a vacation starts, then we have

$$\gamma_{C^C,C}=\lambda_V a^V(n-1)+\beta h(n) \tag{18}$$

Equating γ_{C,C^C} with $\gamma_{C^C,C}$, we have, from 17 and 18,

$$\lambda_V P^V(z)+\beta\Pi^T(z)=\lambda_V P^V(z)z+\beta\Pi^S(z). \tag{19}$$

(It should be noted that, from the PASTA property, we have $P^V(z)=\sum_{n=0}^{\infty}a^V(n)z^n$.) From the definition of λ_V, we have $\lambda_v=\lambda\Pr[Y=0]$. Also, from Little's law, we have $\Pr[Y=0]=\beta E[V]$. Since the number of customers who arrive at the system during a vacation makes the difference between the number of customers at the vacation starting point and that at its ending point, we have $\lambda E[V]=E[N^T]-E[N^S]$. Plugging these results into 19 results in 2.

In a similar way, we derive a relation between queue-length distributions during vacations and those at successive vacation starts and end points, for $M^X/G/1$ vacation queues. Considering possible transitions from C to C^C gives the following transition rate equation:

$$\gamma_{C,C^C}=\lambda_{g,V}a^V(n)+\beta k(n), \tag{20}$$

where $\lambda_{g,V}$ is the rate at which the server is on vacation and customers in batch arrive at the system. Considering possible transitions from C^C to C gives the following transition rate equation:

$$\gamma_{C^C,C}=\lambda_{g,V}\sum_{j=1}^{n}a^V(n-j)g(j)+\beta h(n) \tag{21}$$

Equating γ_{C,C^C} with $\gamma_{C^C,C}$, we have, from 20 and 21,

$$\lambda_{g,V}P^V(z)+\beta\Pi^T(z)=\lambda_{g,V}P^V(z)G(z)+\beta\Pi^S(z) \tag{22}$$

From arguments similar to those below 19, we have

$$P^V(z)=\frac{\Pi^S(z)-\Pi^T(z)}{\lambda_g E[V](1-G(z))}=\frac{\Pi^S(z)-\Pi^T(z)}{g^{-1}E[N^T-N^S](1-G(z))} \tag{23}$$

Remark. 23 corresponds to previous studies on $M^X/G/1$ vacation queues. For the case of exhaustive server vacations,[6] studied an $M^X/G/1$ queue with N-policy and Equation 4.1 in [6] can be rewritten in the form of 23. [7]also studied an $M^X/G/1$ queue with multiple vacations. Equation 11 in [7] can be rewritten in the form of 23. For the case of non-exhaustive server vacations, $M^X/G/1$ retrial queues can be considered. In[8], the equation for $p_0(z)$ on p. 175 and that for $\varphi(z)$ on p. 179 result in a new equation in the form of 23.

Our method can be easily extended to multi-class M/G/1 queues with server vacations. If we let $\lambda_{X,V}$ denote the rate at which a batch consisting of multi-class customers arrives at the system during a server vacation, and let $P^V(\mathbf{z})$ denote the PGF of the customer distribution during vacations, we have

$$\lambda_{X,V}P^V(\mathbf{z})+\beta\Pi^T(\mathbf{z})=\lambda_{X,V}P^V(\mathbf{z})X(\mathbf{z})+\beta\Pi^S(\mathbf{z}). \tag{24}$$

Hence, we have the following relation for multi-class M/G/1 queues with server vacations:

$$P^V(\mathbf{z})=\frac{\Pi^S(\mathbf{z})-\Pi^T(\mathbf{z})}{\lambda_g E[V](1-X(\mathbf{z}))}=\frac{\Pi^{VS}(\mathbf{z})-\Pi^{VT}(\mathbf{z})}{E[X_j]^{-1}E[N_j^{VT}-N_j^{VS}](1-X(\mathbf{z}))}\quad for\ 1\le j\le P, \tag{25}$$

where N_j^S and N_j^T are the number of class-j customers in the system at vacation starting times and the

number of class- j customers at vacation ending times, respectively.

4. Example

Consider a multi-class queue: There are 2 classes of customers. Class- j customers arrive at the system, according to a Poisson process with rate λ_j, $j=1,2$. As soon as the system is empty of customers, the server goes on vacation. The vacation ends as soon as the number of class-1 customers becomes N. ([9]for the waiting time analysis of this queueing system.) Then, the following terms can be easily derived:

$$\Pi^T(\mathbf{z}) = \left(\frac{\lambda_1 z_1}{\lambda_1+\theta}\right)^N \Bigg|_{\theta=\lambda_2-\lambda_2 z_2} = \left(\frac{\lambda_1 z_1}{\lambda-\lambda_2 z_2}\right)^N \tag{26}$$

$$\Pi^S(\mathbf{z}) = 1 \tag{27}$$

$$X(z) = \frac{\lambda_1 z_1 + \lambda_2 z_2}{\lambda} \tag{28}$$

$$E\left[N_1^T - N_1^S\right] = N \tag{29}$$

$$E[X_1] = \lambda_1/\lambda \tag{30}$$

Plugging 26-30 into 25, we can obtain the queue-length distribution during vacations as follows:

$$P^V(\mathbf{z}) = \frac{1-\left(\frac{\lambda_1 z_1}{\lambda-\lambda_2 z_2}\right)^N}{\frac{\lambda}{\lambda_1}N\left(1-\frac{\lambda_1 z_1+\lambda_2 z_2}{\lambda}\right)} = \frac{1-\left(\frac{\lambda_1 z_1}{\lambda-\lambda_2 z_2}\right)^N}{\frac{N}{\lambda_1}\left(\lambda-\lambda_1 z_1+\lambda_2 z_2\right)}. \tag{31}$$

In order to verify 31(31), the marginal queue-length distribution for class-1 customers will be calculated from 31, substituting z_2 with 1. Then, the corresponding PGF is given as follows:

$$P^V(z_1,1) = \frac{1-(z_1)^N}{N(1-z_1)}, \tag{32}$$

which is the PGF of the queue-length distribution for the standard M/G/1 queue with N policy.

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