

C² Rational Cubic Ball Spline Functions

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Abstract

This study deals with the problems of shape preserving curves through positive curves through positive, constrained and convex data. The alternative curve interpolations are developed using a piecewise C² rational cubic Ball spline function with three shape parameters in each subinterval. Simple data dependent conditions are derived for a single shape parameters to preserve the shape of data while the remaining two shape parameters are free used to modify the positive, constrained and convex curves as desired. Graphical results through different numerical examples are presented to show the effectiveness of the proposed schemes.

Keywords: Ball Function, Constrained, Convexity, Interpolation, Positivity

1. Introduction

Spline interpolation is a fundamental tool to visualize the scientific data in Computer Graphics, Computer Engineering and Computer Aided Graphic Design (CAGD). Visualization of scientific data arises from scientific phenomena and complex functions to incorporate the genetic characteristics of data. The basic purpose of data visualization is to present the data information effectively and clearly through graphical means. The representations of scientific data are connected with fields of Engineering, Manufacturing, Military, Education and Medicine etc. There are three basic shape characteristic of data visualization namely positivity, monotonicity and convexity. The objective of present study is to preserve the inherited shape features (positivity, constrained and convexity) of given scientific data.

Positivity is a very important characteristic of shape. The positivity-preserving curve happens in interpolating a physical quantity that cannot be negative which may arise if the data is taken from some scientific, social, or business environments. There are some physical quantities which are always positive. The rate of dissemination of

drugs in the blood, monthly/yearly rainfall data, amount of gas discharge during experiment and percentage mass concentrations in certain chemical reactions, probability distributions, resistance offered by an electric circuit, area, volume and density. Convexity is also a substantial shape characteristic of the data. The significance of the convexity-preserving interpolation problems in industry cannot be denied. Convexity plays significant role in the modeling of cars in automobile industry, aero-plane and ship design because human aesthetic sense always demands curves and surfaces without undulations¹.

Since ordinary cubic spline interpolation generates the unpleasant shapes of data and also it does not preserve the positivity, monotonicity, convexity and constrained-preserving curves. Piecewise cubic Hermite interpolating polynomial (a built-in MATLAB program called PCHIP) has the skill to remove the undesired wiggles but the shape preserving visual model depicts the tight display of data. For shape preserving problem, existing rational cubic schemes used Bernstein basis functions for C¹ and C² continuity of smoothness of interpolation. Due to this, we develop new schemes using ball basis functions which involves three shape parameters and the smoothness of

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the interpolations is C^2 continuity. The schemes deal with positivity, convexity and constrained curves through positive, convex and constrained data, respectively.

Several researchers have focused on the problem of shape preserving curves through scientific data. Some rational cubic spline functions based on Bernstein-Bézier basis functions in the form of (cubic/cubic)²⁻⁶ and (cubic/quadratic)⁷⁻⁹ with shape parameters have been developed to visualize the 2D positive, convex and constrained data. These rational schemes have common characteristics in a way that C^1 and C^2 continuities, local and no extra knots are inserted in the interval where interpolants lost the inherited shape features. In contrast, Hermite piecewise cubic interpolations^{11,12} have been used to solve the problem of interpolating positive and convex data in the sense of positivity and convexity-preserving schemes which were very economical but the methods generally inserted extra knots in the interval to visualize and conserve the shape of data. Sarfraz et al.¹³ considered the piecewise rational cubic spline interpolation to positive, monotone and convex data. The authors introduced two shape parameters in the description of the rational interpolation. Sarfraz et al.¹⁴ developed a piecewise rational cubic/cubic function scheme which involved four shape parameters in each subinterval. Data dependent constraint were derived for two shape parameters to preserve the shape of data while the other two shape parameters left free for modification of the positivity, convexity and constrained curve.

An alternative curve schemes using piecewise rational cubic ball spline function are developed in this paper to guarantee C^2 continuity. Three shape parameters u_i , v_i and w_i are used for the description of rational cubic function to preserve the shape of the data. Two shape parameters u_i and v_i are free parameters which can be used to modify the shape of the curve and one shape parameter w_i automatically to preserve the shape of positive, constrained and convex data. This work is an improvement in the existing rational schemes. The new schemes have the following valuable and advantageous improvements.

- i. They provide a C^2 continuity, while the schemes^{2,4,6,10-13}, the degree of smoothness is C^1 .
- ii. They preserve the shape of data without inserting the extra knots while the schemes^{11,12} are needed extra knots in the subinterval where they do not preserve the shape of data.
- iii. It is concluded from experimental results the present schemes are flexible, simple, local and economical.

- iv. It is found that shape preserving curves are visually pleasing and smooth as compared to the existing schemes in^{3,5,7,8,13,14}.
- v. They work for uniform and non-uniform spaced data while the scheme in¹⁵ only works for uniform spaced data.
- vi. The derivative parameters d_i can be calculated in this paper using tri-diagonal single system of linear equations is much more efficient than, for instance, solving the three systems of linear equations in¹⁶.
- vii. Designers are allowed in the present study to modify the curve by introducing free parameters which can be used freely to produce a visually better pleasant curve but in¹⁴ the authors proposed scheme does not allow the designer to modify the shape of curves.
- viii. The proposed curve schemes are unique in their representation and they are equally applicable for the data with derivative or without derivatives.
- ix. The authors in¹⁷ assumed certain function values and derivative values to control the shape of the data while in this work, simple data dependent conditions for single shape parameters are derived to attain a positivity, constrained and convexity-preserving curves through positive, constrained and convex data, respectively.

The remainder of the paper is organized as follows. The second section introduces a C^2 rational cubic ball spline function and determination of derivatives is discussed in the third section. The fourth section discusses rational cubic ball spline function in terms of positivity-preserving curve interpolation. The fifth section deals with the problem of constrained curve interpolation. The six section deals with problem of convexity-preserving curve interpolation and how to generate C^2 piecewise interpolants. The outputs obtained from several test problems are presented in the seventh section. Finally, in section 8, the conclusion of this study is given.

2. C^2 Rational Cubic Ball Spline Function

Rational Ball spline interpolations are more accurate than polynomial spline functions and rational Bernstein-Bézier interpolations as it has much wider range of shapes, simple form, less oscillatory and to model complex structure with a fairly low degree in both the numerator and denominator. In this section, we rewrite the rational cubic

Ball spline function developed by Jafaar et al⁹. Let $\{(x_p, f_p), i = 0, 1, 2, \dots, n\}$ be a given set of data points and defined over the interval $[a, b]$ where $a = x_0 < \dots < x_n = b$. Let $h_i = x_{i+1} - x_i$, $\Delta_i = (f_{i+1} - f_i)/h_i$, $i = 1, 2, \dots, n-1$. A piecewise rational cubic Ball spline function is defined over each interval $I_i = [x_p, x_{i+1}]$, $i = 0, 1, 2, \dots, n-1$, as:

$$Z(x) = Z_i(\eta) = \frac{p_i(\eta)}{q_i(\eta)} \quad (1.1)$$

where

$$p_i(\eta) = \varepsilon_0(1-\eta)^2 + \varepsilon_1\eta(1-\eta) + \varepsilon_2\eta^2(1-\eta) + \varepsilon_3\eta^2$$

$$q_i(\eta) = u_i(1-\eta)^2 + a_i\eta(1-\eta) + \beta_i\eta^2(1-\eta) + v_i\eta^2$$

with $\eta = (x - x_i)/h_i$, $\eta \in [0, 1]$. Based on (1.1), the numerator and denominator are the cubic Ball polynomial and the denominator is nonzero with u_i , v_i , a_i and β_i are nonzero shape parameters. To make rational cubic Ball function with four shape parameters, equation (1.1) become rational cubic Ball function with three shape parameters and the middle values of a_i and β_i can be assumed as $a_i = \beta_i = u_i + v_i + w_i \geq 0$, $i = 0, 1, 2, \dots, n-1$. A rational cubic Ball function $Z(x)$, with three shape parameters can be written as

$$Z(x) = Z_i(\eta) = \frac{p_i(\eta)}{q_i(\eta)} = \frac{\varepsilon_0(1-\eta)^2 + \varepsilon_1\eta(1-\eta) + \varepsilon_2\eta^2(1-\eta) + \varepsilon_3\eta^2}{u_i(1-\eta)^2 + (u_i + v_i + w_i)(1-\eta) + v_i\eta^2} \quad (1.2)$$

The two shape parameters u_i and v_i in equation (1.2) are set as a free shape parameters which provides a freedom to modify the curve as desired while shape parameters w_i is a constraint to preserve the shape of data. To achieve C^1 continuity, a rational cubic Ball function (1.2) must satisfies the following conditions

$$\begin{cases} Z(x_i) = f_i, & Z(x_{i+1}) = f_{i+1} \\ Z'(x_i) = d_i, & Z'(x_{i+1}) = d_{i+1} \end{cases} \quad (1.3)$$

where $Z'(x)$ represents the first derivatives with respect x and d_i denote the values of derivative (tangents) at the knot. The results provide the following coefficients of piecewise cubic Ball interpolant (1.2)

$$\begin{aligned} \varepsilon_0 &= u_i f_i \\ \varepsilon_1 &= (u_i + v_i + w_i) f_i + u_i h_i d_i \\ \varepsilon_2 &= (u_i + v_i + w_i) f_{i+1} - v_i h_i d_{i+1} \\ \varepsilon_3 &= v_i f_{i+1} \end{aligned} \quad (1.4)$$

To ensure rational cubic Ball function (1.2) is C^2 continuity, the second derivative continuity need to impose at the knot respect to x_i .

$$Z''(x_{i+}) = Z''(x_{i-}), i = 1, 2, \dots, n-1 \quad (1.5)$$

where $Z''(x)$ denotes the second derivatives with respective to x and from (1.5), a piecewise cubic Ball interpolant $Z(x_i) \in C^2[x_p, x_{i+1}]$ provides the following tri-diagonal system of linear equations.

$$\mu_i d_{i-1} + \kappa_i d_i + \eta_i d_{i+1} = \zeta_i \quad (1.6)$$

Where

$$\begin{aligned} \mu_i &= u_{i-1} u_i h_i \\ \kappa_i &= (u_{i-1} + v_{i-1} + w_{i-1}) u_i h_i + (u_i + v_i + w_i) v_{i-1} h_{i-1} \\ \eta_i &= v_{i-1} v_i h_{i-1} \\ \zeta_i &= (2u_{i-1} + v_{i-1} + w_{i-1}) \Delta_{i-1} u_i h_i + (u_i + 2v_i + w_i) \Delta_i v_{i-1} h_{i-1} \end{aligned} \quad (1.7)$$

Remark 1: When the values of shape parameters where $u_i = v_i = 1$ and $w_i = 0$ in each subinterval $I_i = [x_p, x_{i+1}]$, $i = 0, 1, 2, \dots, n-1$. The rational becomes the non-rational cubic Ball function like standard cubic Hermite Spline function¹⁹.

3. Determination of Derivatives

In scientific applications, the derivative values are not given directly and the derivatives d_1 and d_n can be calculated from the given data points by using Arithmetic Mean Method. The Arithmetic Mean Method will be described in details as follows:

$$d_1 = \begin{cases} 0, & \text{if } \Delta_1 = 0 \text{ or sign } (d_1) \neq \text{sign } (\Delta_1) \\ \Delta_1 + \frac{(\Delta_1 - \Delta_2)h_1}{h_1 + h_2}, & \text{otherwise} \end{cases}$$

$$d_n = \begin{cases} 0, & \text{if } \Delta_{n-1} = 0 \text{ or sign } (d_n) \neq \text{sign } (\Delta_{n-1}) \\ \Delta_{n-1} + \frac{(\Delta_{n-1} - \Delta_{n-2})h_{n-1}}{h_{n-1} + h_{n-2}}, & \text{otherwise} \end{cases} \quad (1.8)$$

The tri-diagonal system of linear equation (1.6) developed $(n-2)$ linear equations and constructed the values of unknown derivative parameters d_i , $i = 2, \dots, n-1$. To estimate values of derivative parameter d_i . Therefore LU-decomposition method is implemented in the system of linear equation.

4. Positivity-Preserving C^2 Rational Cubic Ball Interpolation

The present study deals with the problem of shape preserving data visualization through regular positive data. A proposed scheme rational cubic Ball function

with cubic denominator with three shape parameters is presented to generate positive curves through positive data. Let $\{(x_i, f_i), i = 0, 1, 2, \dots, n\}$ be a given set of positive data points such that $x_0 < x_1 < x_2 < \dots < x_n$ and $f_i > 0, i = 1, 2, \dots, n$. A piecewise rational cubic Ball function (1.2) preserves the positivity-preserving curves through given positive data if

$$Z(x) = Z_i(\eta) = \frac{p_i(\eta)}{q_i(\eta)} > 0 \quad (1.9)$$

The rational cubic Ball function (1.2) preserves the positivity-preserving curve of positive data if both $p_i(\eta) > 0$ and $q_i(\eta) > 0$. As the necessary conditions $u_i, v_i > 0$ and $w_i > 0$ are enough to guarantee denominator $q_i(\eta) > 0$. Since $u_i, v_i > 0$ then $q_i(\eta)$ should be positive.

According to Schmidt and HeB¹⁸, the cubic polynomial $p_i(\eta) > 0$ if and only if

$$(p'_i(0), p'_i(1)) \in R_1 \cup R_2 \quad (1.10)$$

where

$$R_1 = \left\{ (a, b) : a > \frac{-3p(0)}{h_i}, b < \frac{3p(1)}{h_i} \right\}$$

$$R_2 = \left\{ (a, b) : 36f_i f_{i+1} (a^2 + b^2 + ab - 3\Delta_i(a+b) + 3\Delta_i^2) + 4h_i (a^3 f_{i+1} - b^3 f_i) + 3(af_{i+1} - bf_i)(2h_i ab - 3af_{i+1} + 3bf_i) - h_i^2 a^2 b^2 \geq 0 \right\} \quad (1.11)$$

The following values

$$p'_i(0) = \frac{(-u_i + v_i + w_i)f_i + u_i h_i d_i}{h_i}$$

$$p'_i(1) = \frac{-(u_i - v_i + w_i)f_{i+1} + v_i h_i d_{i+1}}{h_i} \quad (1.12)$$

Now $(p'_i(0), p'_i(1)) \in R_1$ if

$$p'_i(0) > \frac{-3u_i f_i}{h_i}$$

$$\frac{u_i h_i d_i + (-u + v + w)f_i}{h_i} > \frac{-3u_i f_i}{h_i} \quad (1.13)$$

and

$$p'_i(1) < \frac{-3v_i f_{i+1}}{h_i}$$

$$\frac{-(u_i - v_i + w_i)f_{i+1} + v_i h_i d_{i+1}}{h_i} < \frac{-3v_i f_{i+1}}{h_i} \quad (1.14)$$

From equations (1.13) and (1.14), the constraints for the shape parameters w_i are as follows:

$$\begin{cases} w_i > -\left((2u_i + v_i) + \frac{u_i h_i d_i}{f_i} \right) \\ w_i > -\left((u_i + 2v_i) - \frac{v_i h_i d_{i+1}}{f_{i+1}} \right) \end{cases} \quad (1.15)$$

Further, $(p'_i(0), p'_i(1)) \in R_2$ if

$$\begin{cases} 36f_i f_{i+1} (a^2 + b^2 + ab - 3\Delta_i(a+b) + 3\Delta_i^2) \\ + 4h_i (a^3 f_{i+1} - b^3 f_i) + 3(af_{i+1} - bf_i) \\ (2h_i ab - 3af_{i+1} + 3bf_i) - h_i^2 a^2 b^2 > 0 \end{cases} \quad (1.16)$$

where $a = p'_i(0)$ and $b = p'_i(1)$.

Remark 2: The constraints of shape parameters w_i can be obtained from equation (1.16) but they require a lot of computations and since it is not a convenient choice, the alternate choice need to be considered, which is the conditions in equation (1.13) and (1.14). The above results can be written as follows:

Theorem 1: The rational cubic Ball interpolant (1.2) preserves the positivity-preserving curve through positive data in each subinterval $[x_i, x_{i+1}]$ if the following constraints are satisfied.

$$\begin{cases} u_i > 0, v_i > 0 \\ w_i > \max \left\{ 0, -\left((2u_i + v_i) + \frac{u_i h_i d_i}{f_i} \right), -\left((u_i + 2v_i) - \frac{v_i h_i d_{i+1}}{f_{i+1}} \right) \right\} \end{cases}$$

The above result can be rewritten as follows:

$$\begin{cases} u_i > 0, v_i > 0 \\ w_i = r_i + \max \left\{ 0, -\left((2u_i + v_i) + \frac{u_i h_i d_i}{f_i} \right), -\left((u_i + 2v_i) - \frac{v_i h_i d_{i+1}}{f_{i+1}} \right) \right\}, r_i > 0. \end{cases}$$

Algorithm 1

- (i) Given a positive data set $\{(x_i, f_i), i = 0, 1, 2, \dots, n\}$.
- (ii) For $i = 0, 1, 2, \dots, n$.
 - (i) Calculate the values of h_i .
 - (ii) Calculate the values of Δ_i .
 - (iii) Approximate the end points derivatives d_1 and d_n using Arithmetic Mean Method.
 - (iv) Approximate the derivative values d_i for $i = 2, \dots, n-1$ using LU-decomposition methods.

- (v) Choose the any positive value for free shape parameters u_i and v_i and calculate the values of constrained parameter w_i using Theorem 1.
- (vi) Insert all calculated values from (i)-(v) in the proposed rational cubic Ball spline function (1.2) to preserve the C^2 positivity-preserving curve through positive data.

5. Constrained-Preserving C^2 Rational Cubic Ball Interpolation

In this section, we discuss the problem of shape preserving curve through constrained data using C^2 rational cubic Ball spline with three shape parameters. We consider constrained data that lies above the straight line. It is required to determine the most suitable values of constraints for the shape parameters to preserve the shape of the curve that lies above the straight line. Let $\{(x_i, f_i), i = 0, 1, 2, \dots, n\}$ be a given set of data points that are lying above the straight line $y = mx + c$.

So

$$f_i > mx_i + c, \quad \forall i = 0, 1, 2, \dots, n \quad (1.17)$$

Where m and c are defined as the slope and y -intercept of the line respectively. The curve lies above the straight line if the C^2 Cubic Ball function (1.2) satisfies the following condition:

$$Z(x) > mx + c, \quad \forall x \in [x_0, x_n] \quad (1.18)$$

For each subinterval $I_i = [x_i, x_{i+1}]$, the equation (1.18) can be written as:

$$Z(x) = Z_i(\eta) = \frac{p_i(\eta)}{q_i(\eta)} > mx_i + c \quad (1.19)$$

The equation of straight line in parameter η is defined as

$$r_i(1 - \eta) + s_i\eta, \quad \eta \in [0, 1] \quad (1.20)$$

where

$$r_i = mx_i + c \text{ and } s_i = mx_{i+1} + c.$$

The parametric form of the equation (1.19) is written as

$$Z(x) = Z_i(\eta) = \frac{p_i(\eta)}{q_i(\eta)} > r_i(1 - \eta) + s_i\eta, \quad i = 0, 1, 2, \dots, n.$$

or

$$Z(x) = Z_i(\eta) = \frac{p_i(\eta)}{q_i(\eta)} - (r_i(1 - \eta) + s_i\eta) > 0 \quad (1.21)$$

Multiply the nonzero denominator $q_i(\eta)$ to both sides of equation (1.21), we have

$$F_i(x) = (1 - \eta)^3 G_0 + (1 - \eta)^2 G_1 + (1 - \eta) \eta G_2 + (1 - \eta) \eta^2 G_3 + \eta^2 G_4 + \eta^3 G_5 \quad (1.22)$$

with

$$\begin{aligned} G_0 &= -u_i r_i \\ G_1 &= u_i f_i \\ G_2 &= (u_i + v_i)(f_i - r_i) + w_i(f_i - r_i) - u_i(s_i - d_i h_i) \\ G_3 &= (u_i + v_i)(f_{i+1} - s_i) + w_i(f_{i+1} - s_i) - v_i(r_i + h_i d_{i+1}) \\ G_4 &= v_i f_{i+1} \\ G_5 &= -v_i s_i \end{aligned} \quad (1.23)$$

The necessary conditions for constrained-preserving curve are

$$\begin{aligned} (f_i - r_i) &> 0 \\ (f_{i+1} - s_i) &> 0 \end{aligned} \quad (1.24)$$

From the conditions in (1.24), it is clear that $G_2 > 0$ and $G_3 > 0$.

$$G_2 > 0 \text{ if } w_i > \frac{u_i(s_i - h_i d_i)}{f_i - r_i} \quad (1.25)$$

$$G_3 > 0 \text{ if } w_i > \frac{v_i(r_i + h_i d_{i+1})}{f_{i+1} - s_i} \quad (1.26)$$

The above conditions can be summarized as

$$\begin{cases} u_i > 0, v_i > 0 \\ w_i > \max \left\{ 0, \frac{u_i(s_i - h_i d_i)}{f_i - r_i}, \frac{v_i(r_i + h_i d_{i+1})}{f_{i+1} - s_i} \right\} \end{cases} \quad (1.27)$$

The above results can be written as:

$$\begin{cases} u_i > 0, v_i > 0 \\ w_i = l_i + \max \left\{ 0, \frac{u_i(s_i - h_i d_i)}{f_i - r_i}, \frac{v_i(r_i + h_i d_{i+1})}{f_{i+1} - s_i} \right\}, l_i > 0 \end{cases} \quad (1.28)$$

Theorem 2: A rational cubic Ball spline function (1.2) preserves the shape of the C^2 constrained curve that lies above the straight line over the interval $[x_i, x_{i+1}]$ if and only if the shape parameters u_i , v_i and w_i satisfy the conditions given in (1.28).

Algorithm 2

- (i) Given a constrained data set $\{(x_i, f_i), i = 0, 1, 2, \dots, n\}$ that is lying above the line $y = mx + c$.
- (ii) Repeat steps (ii)-(iv) of Algorithm 1 for the approximation of h_i, Δ_i and $d_i, d_{i+1}, d_i, d_{i+1}, i = 2, \dots, n-1$.
- (iii) Choose the any positive value for free shape parameters u_i and v_i and calculate the values of constrained parameter w_i using Theorem 2.

Insert all calculated values from (i)-(iii) in the proposed rational cubic Ball spline function (1.2) to preserve the C² constrained curve through constrained data.

6. Convexity-Preserving C² Rational Cubic Ball Interpolation

The C² rational cubic Ball spline function (1.2) with shape parameters does not preserve the shape of convex data by inserting the manually values of shape parameters. To overcome this problem, simple sufficient conditions are derived for single shape parameters to preserve the convexity of convex data in this section. Some mathematical treatment is required to show the effectiveness of the proposed scheme.

Let $\{(x_i, f_i), i = 0, 1, 2, \dots, n\}$ be a given set of convex data point. For the strictly convex set of data

$$\Delta_1 < \Delta_2 < \Delta_3 < \dots < \Delta_{n-1} \quad (1.29)$$

Correspondingly for concave set of data,

$$\Delta_1 > \Delta_2 > \Delta_3 > \dots > \Delta_{n-1} \quad (1.30)$$

The necessary conditions on derivative parameters d_i for a convex interpolation $Z_i(\eta)$ can be presented as follows:

$$d_1 < \Delta_1 < \dots < \Delta_{i-1} < d_i < \Delta_i < \dots < \Delta_{n-1} < d_n \quad (1.31)$$

Similarly, for concave interpolation

$$d_1 > \Delta_1 > \dots > \Delta_{i-1} > d_i > \Delta_i > \dots > \Delta_{n-1} > d_n \quad (1.32)$$

Based on the equation (1.31), the necessary conditions for the convexity of data are

$$\begin{aligned} \Delta_i - d_i &\geq 0 \\ d_{i+1} - \Delta_i &\geq 0 \\ d_{i+1} - d_i &\geq 0 \end{aligned} \quad (1.33)$$

The C² piecewise rational cubic Ball interpolant $Z_i(x)$ is convex if and only if $Z_i''(x) > 0, \forall x \in [x_i, x_{i+1}]$. The $Z_i''(x)$ is

$$Z_i''(x) = \frac{\left[(1-\eta)^4 B_0 + (1-\eta)^4 \eta B_1 + (1-\eta)^3 \eta^2 B_2 + (1-\eta)^2 \eta^3 B_3 + (1-\eta) \eta^4 B_4 + \eta^4 B_5 \right]}{h_i(q(\eta))^3} \quad (1.34)$$

with

$$\begin{aligned} B_0 &= 2u_i^2[u_i(\Delta_i - d_i) + w_i(\Delta_i - d_i) + v_i(\Delta_i - d_i) - v_i(d_{i+1} - \Delta_i)] \\ B_1 &= B_0 + 2u_i^2[3v_i(\Delta_i - d_i)] \\ B_2 &= B_0 + 6u_i[2u_i v_i(\Delta_i - d_i) + v_i^2(d_{i+1} - \Delta_i)] \\ B_3 &= B_0 + 6v_i[2u_i v_i(d_{i+1} - \Delta_i) + u_i^2(\Delta_i - d_i)] \\ B_4 &= B_0 + 2v_i^2[3u_i(d_{i+1} - \Delta_i)] \\ B_5 &= 2v_i^2[v_i(d_{i+1} - \Delta_i) + w_i(d_{i+1} - \Delta_i) - u_i(\Delta_i - d_i) + u_i(d_{i+1} - \Delta_i)] \end{aligned} \quad (1.35)$$

The $Z_i''(x) > 0$ if both $B_i > 0, i = 0, 1, 2, \dots, 5$ and $h_i(q(\eta)) > 0$. Since $u_i, v_i, w_i \geq 0$ so denominator in (1.34) should be positive. After some simplification, the sufficient conditions based on $B_i > 0, i = 0, 1, 2, \dots, 5$ if

$$\left\{ \begin{aligned} &u_i > 0, v_i > 0 \\ &w_i > \max \left\{ 0, \frac{u_i(\Delta_i - d_i) - (u_i + v_i)(d_{i+1} - \Delta_i)}{d_{i+1} - \Delta_i}, \frac{v_i(d_{i+1} - \Delta_i) - (u_i + v_i)(\Delta_i - d_i)}{\Delta_i - d_i} \right\} \end{aligned} \right\} \quad (1.36)$$

The sufficient conditions in (1.36) for convexity can be rewritten as

$$\left\{ \begin{aligned} &u_i > 0, v_i > 0 \\ &w_i = k_i + \max \left\{ 0, \frac{u_i(\Delta_i - d_i) - (u_i + v_i)(d_{i+1} - \Delta_i)}{d_{i+1} - \Delta_i}, \frac{v_i(d_{i+1} - \Delta_i) - (u_i + v_i)(\Delta_i - d_i)}{\Delta_i - d_i} \right\}, k_i > 0. \end{aligned} \right\} \quad (1.37)$$

The above result can be summarized as follows:

Theorem 3: The C² rational cubic Ball scheme preserves the convexity of convex data in each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, n$ involving free shape parameters u_i and v_i that satisfy the following sufficient conditions.

$$\left\{ \begin{aligned} &u_i, v_i > 0 \\ &w_i = k_i + \max \left\{ 0, \frac{u_i(\Delta_i - d_i) - (u_i + v_i)(d_{i+1} - \Delta_i)}{d_{i+1} - \Delta_i}, \frac{v_i(d_{i+1} - \Delta_i) - (u_i + v_i)(\Delta_i - d_i)}{\Delta_i - d_i} \right\}, k_i > 0. \end{aligned} \right\}$$

Algorithm 3

- (i) Given a convex data set $\{(x_i, f_i), i = 0, 1, 2, \dots, n\}$.
- (ii) Repeat steps (ii)-(iv) of Algorithm 1 for the approximation of h_i, Δ_i and $d_i, d_n, d_p, i = 2, \dots, n - 1$.
- (iii) Choose the any positive value for free shape parameters u_i and v_i and calculate the values of constrained parameter w_i using Theorem 3.

Put all calculated values from (i)-(iii) in the proposed rational cubic Ball spline function (1.2) to preserve the C^2 convexity-preserving curve through convex data.

7. Numerical Examples

A C^2 rational cubic Ball spline interpolations for positive, constrained and convex data are used in this section to preserve the positivity, constrained and convexity-preserving curves. Several test problems are presented in this section to exhibit the capability and effectiveness of the proposed positive, constrained and convex rational cubic Ball spline interpolations. A desired smooth and visually pleasing positive, constrained and convex curves obtained from several examples are compared with PCHIP and existing Bernstein- Bezier rational schemes^{1-8,11-14} are also part of this section.

Example 1: The positive data set taken in Table 1 is borrowed from¹³. The data was collected from a chemical experiment when the molar volume of the gas (in l/mol) at different temperature codes was observed. The x -values are temperature code and the f -values are the molar volume of the gas. Based on this positive data, Figure 1 is generated by non-rational cubic Ball scheme¹⁹. The scheme does not preserve the shape of curve. Figure 2 is produced by using Piecewise Cubic Hermite polynomial Interpolation (PCHIP) to preserve the shape of curve

Table 1. A positive set of chemical experiment data

i	x_i	f_i
1	1	24.616
2	2	2.461
3	3	41.027
4	5	4.102
5	7	57.437
6	8	5.7438
7	9	0.5744

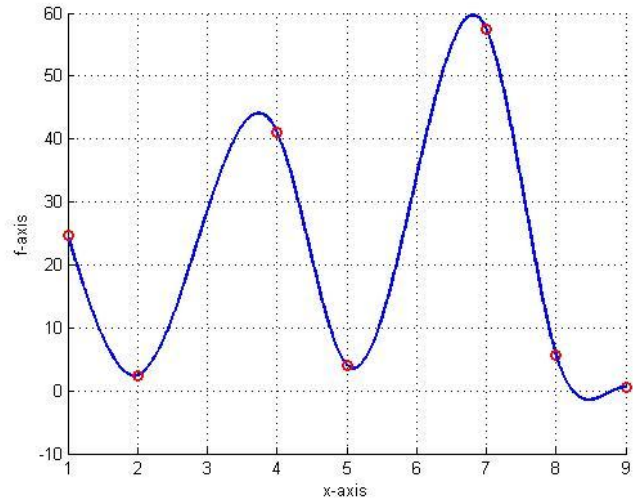


Figure 1. Non-rational cubic Ball curve.

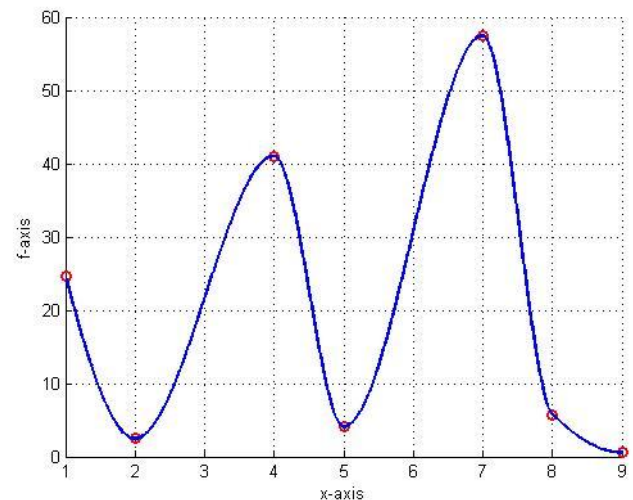


Figure 2. PCHIP curve.

but the curve looks a bit tight compared to Figure 3 and Figure 4. Figure 3 and Figure 4 are generated by proposed positivity-preserving rational cubic Ball interpolant. The scheme is preserved the shape of curve of positive data with different values of shape parameters. The numerical results and graphical view show that both Figure 3 and Figure 4 looks more nice and smooth as compared to Figure 2.

Example 2: The data set given in Table 2 has been taken from³. A non-positive curve based on positive data is drawn in Figure 5 by using a non-rational cubic Ball spline scheme¹⁹. Figure 6 is generated by PCHIP. Figure 7 and Figure 8 are generated by using positive rational cubic Ball spline interpolation. The effectiveness of the devel-

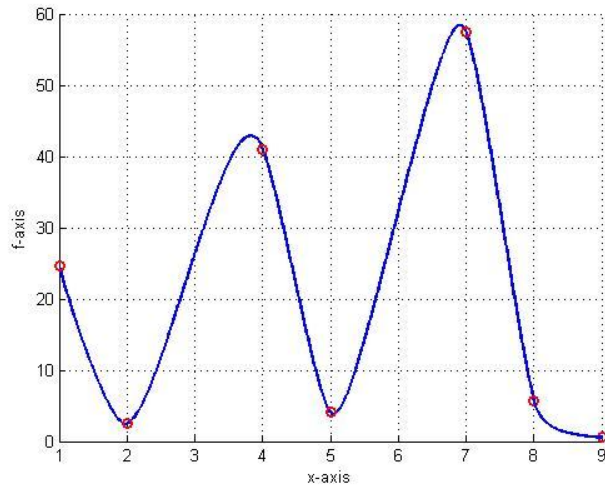


Figure 3. Positivity-preserving curve using C^2 rational cubic Ball spline function $u_i = v_i = 0.05$.

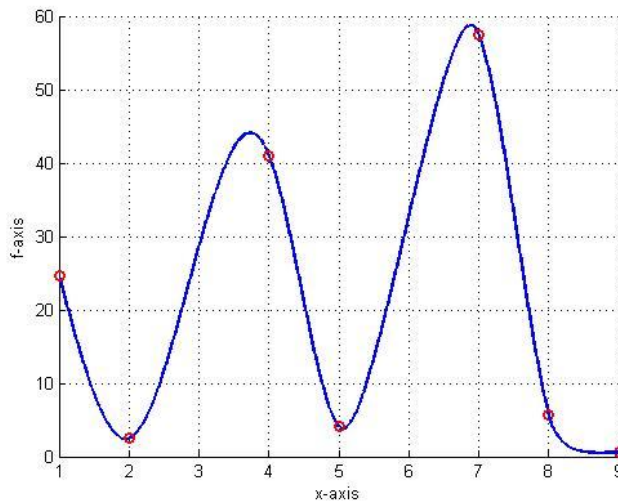


Figure 4. Positivity-preserving curve using C^2 rational cubic Ball spline function $u_i = v_i = 2.5$.

Table 2. A positive wind velocity data

i	x_i	f_i
1	0	2.00
2	0.25	0.60
3	0.50	0.10
4	1.00	0.13
5	1.50	1.00
6	2.00	0.50
7	2.50	1.10
8	3.00	0.25
9	4.00	0.20

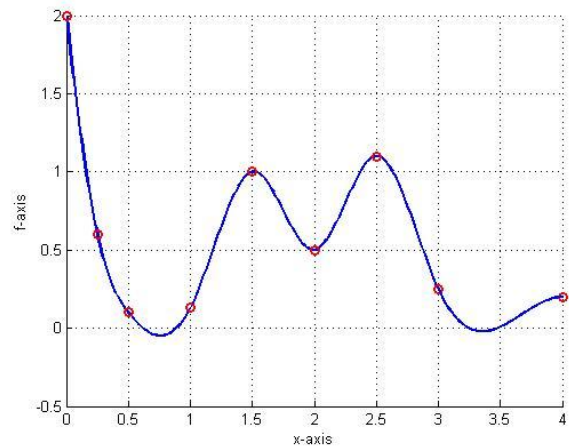


Figure 5. Non-rational cubic Ball curve.

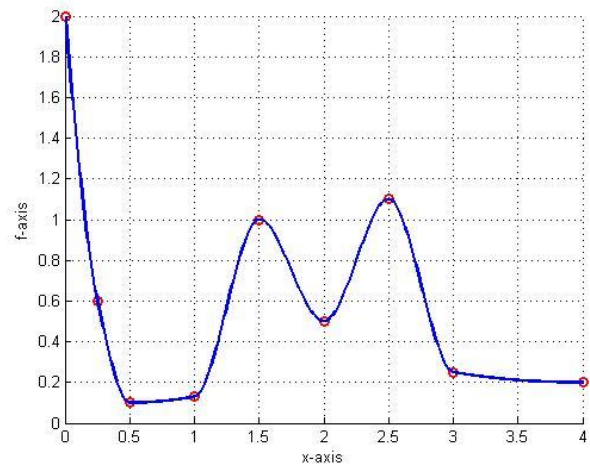


Figure 6. PCHIP curve.

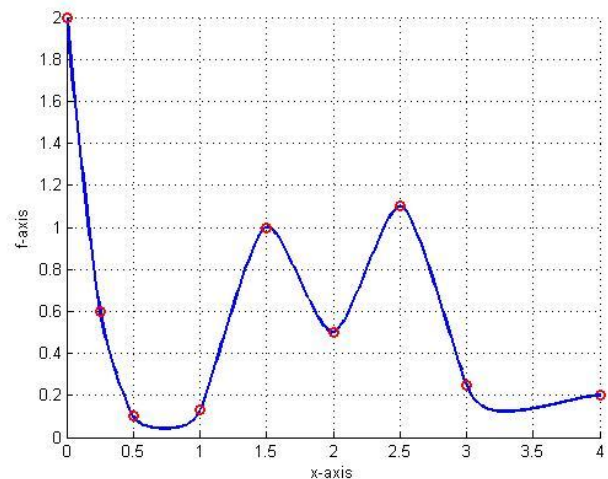


Figure 7. Positivity-preserving curve using C^2 rational cubic Ball spline function $u_i = v_i = 0.5$.

oped scheme with different values of shape parameters can be seen in Figure 7 and Figure 8.

Example 3: The constrained data set is taken in Table 3 is borrowed from⁹. Figure 9 is produced by non-rational cubic Ball spline scheme when $u_i = v_i = 1$ and $w_i = 0$. It does not preserve the shape of constrained data. Fig. 10 is drawn by using PCHIP to preserve the shape of curve that lies above the straight line $y_i = 0.07x_i + 0.15$ but it does not look smooth curve. Figure 11 is generated by constrained rational cubic Ball interpolation to preserve the shape of the curve that lies above the straight line. The effect of shape parameters can be seen in Figure 9, Figure 10 and Figure 11.

Example 4: In Table 4, a constrained data set being used which lies above the straight line $y_i = 0.5x_i + 0.15$. In Figure. 12, the curve is not preserve the shape of constrained data that is lying below the straight line when $u_i = v_i = 1$ and $w_i = 0$. Figure 13 is drawn with help of PCHIP to preserve

Table 3. A constrained data set

i	x_i	f_i
1	0	5.5
2	2	5.5
3	3	5.5
4	5	5.5
5	6	16
6	8	50
7	11	60
8	12	80

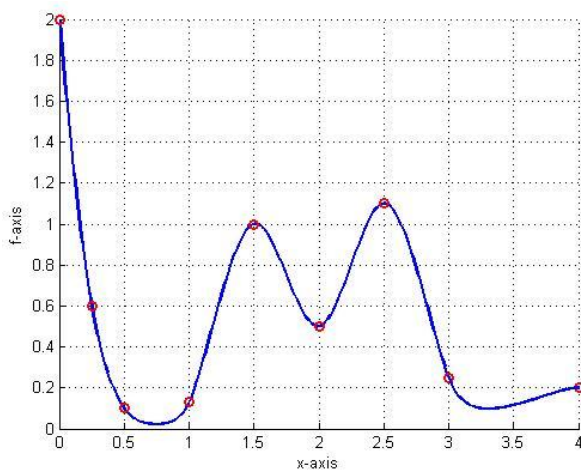


Figure 8. Positivity-preserving curve using C^2 rational cubic Ball spline function $u_i = v_i = 2.5$.

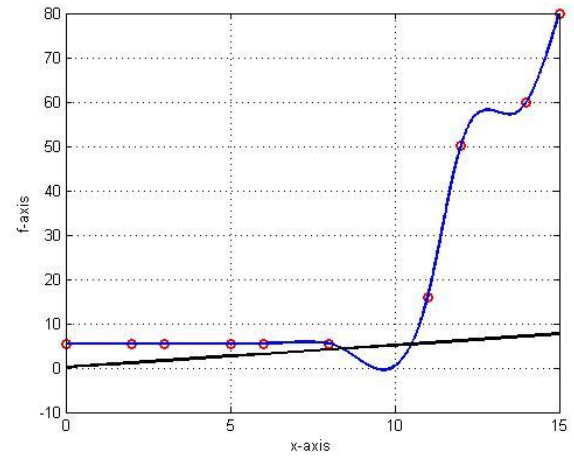


Figure 9. Non-rational cubic Ball curve.

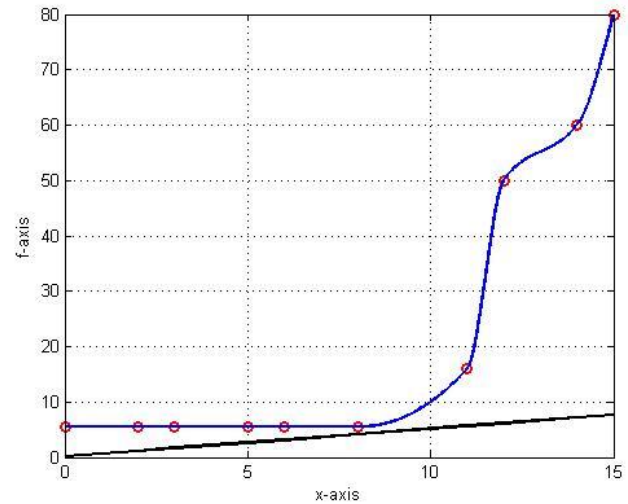


Figure 10. PCHIP curve.

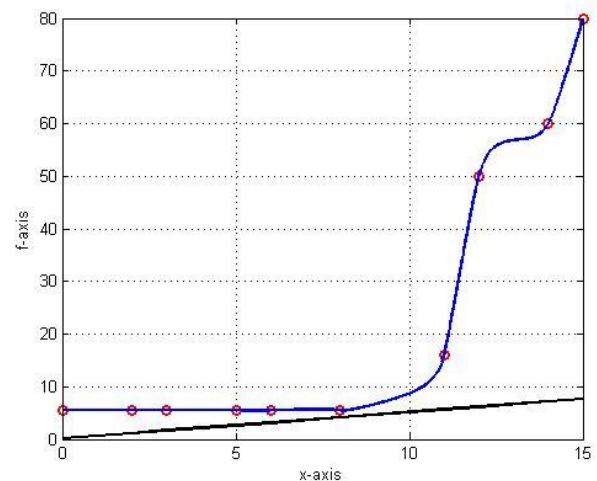
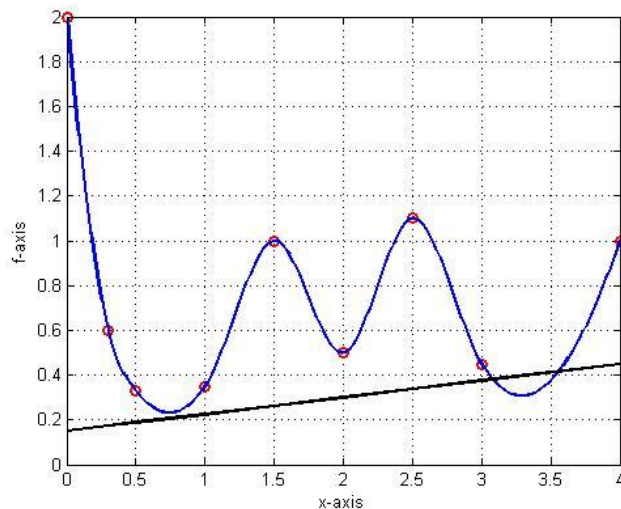
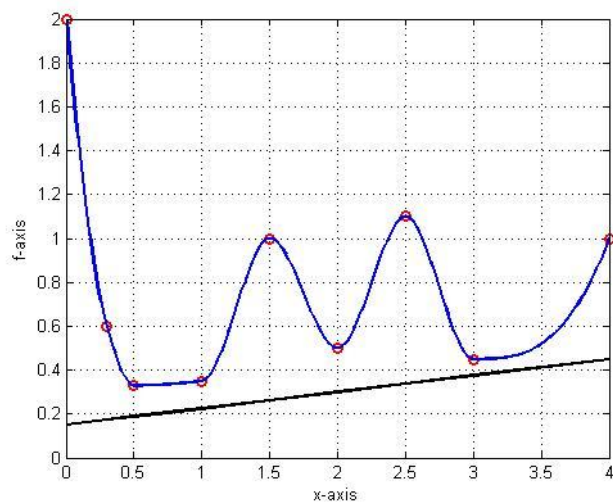


Figure 11. Constrained curve using C^2 rational cubic Ball with $u_i = v_i = 0.25$.

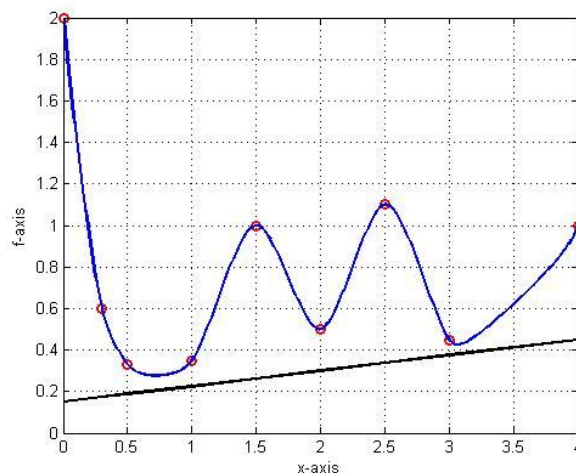
Table 4. A constrained data set

i	x_i	f_i
1	0	2.00
2	0.30	0.60
3	0.50	0.33
4	1.00	0.35
5	1.50	1.00
6	2.00	0.50
7	2.50	1.10
8	3.00	0.45
9	4.00	1.00

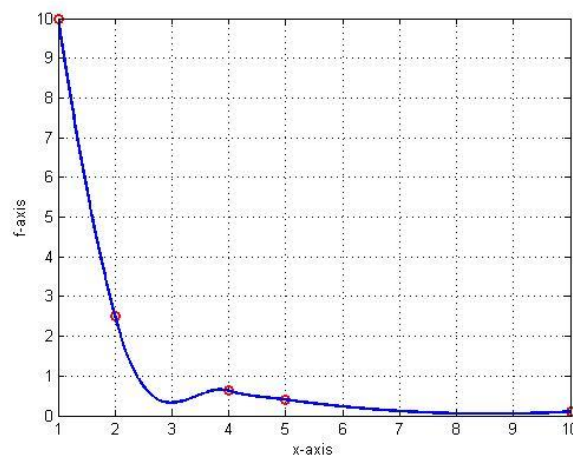
**Figure 12.** Non-rational cubic Ball curve.**Figure 13.** PCHIP curve.

the shape of constrained data that is lying above the straight line but its display looks tight. Figure 14 is generated by using constrained rational cubic Ball spline interpolation with the values of shape parameters $u_i = v_i = 0.05$.

Example 5: The convex data given in Table 5. Figure 15 is an initial default curve generated by non-rational Cubic

**Figure 14.** Constrained-preserving curve using rational cubic Ball spline function which lies above the straight line when $u_i = v_i = 0.05$.**Table 5** A convex data set

i	x_i	f_i
1	1	10.00
2	2	2.500
3	4	0.625
4	5	0.400
5	10	0.100

**Figure 15.** Non-rational cubic Ball curve.

Ball spline scheme and it does not preserve the local convexity of convex data. Figure 16 is generated by Piecewise Cubic Interpolating Polynomial (PCHIP), the curve preserves the convexity but it does not look smooth at certain points. Figure 17 and Figure 18 are generated by local convexity-preserving curve interpolation with the values of free parameters $u_i = v_i = 0.002$ and $u_i = v_i = 3$ respectively. The effect of shape parameters can be seen by noting the difference in smoothness of the curves in Figure 16, Figure 17 and Figure 18.

Example 6: The convex data set taken in Table 6 is borrowed from⁷. Figure 19 is generated by non-rational cubic Ball spline scheme¹⁹ which does not preserve the convexity of convex data. Figure 20 is generated by Piecewise Cubic Interpolating Polynomial (PCHIP) which it preserves the convex curve but at certain interval it does not look smooth. Figure 21 and Figure 22 are generated by

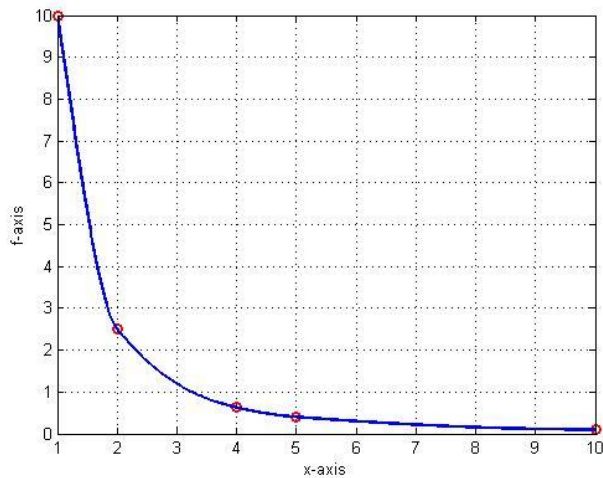


Figure 16. PCHIP curve.

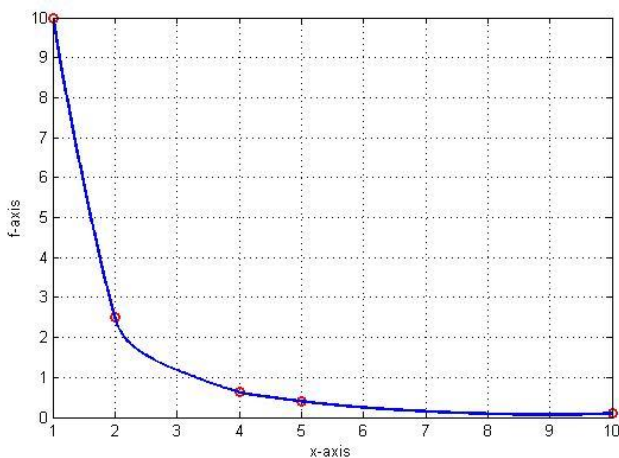


Figure 17. C^2 convexity-preserving curve.

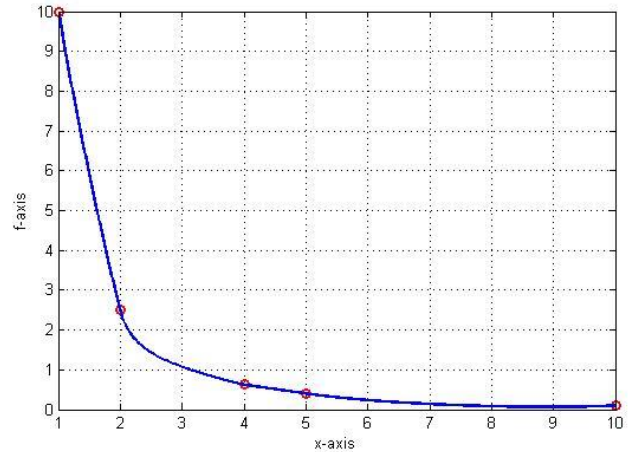


Figure 18. Convex curve using C^2 rational cubic Ball function.

Table 6. A convex data set

i	x_i	f_i
1	-8	4.50
2	-7	4.00
3	2.2	3.55
4	7	4.00
5	10	4.50
6	12	5.00

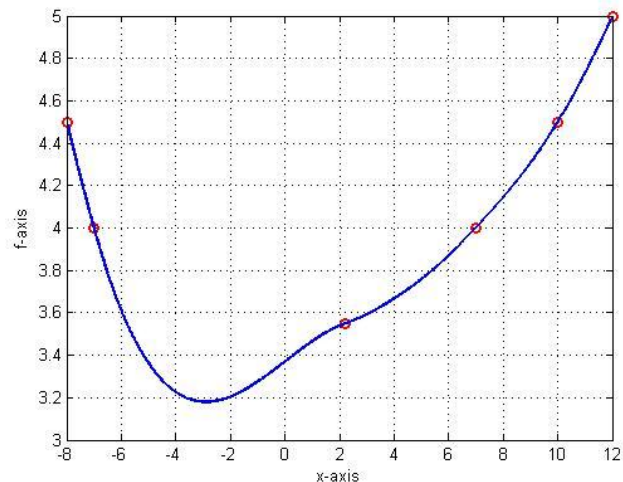


Figure 19. Non-rational cubic Ball curve.

local convexity-preserving C^2 rational cubic Ball interpolation with the values of free shape parameters $u_i = v_i = 0.0005$ and $u_i = v_i = 2$. The effect of shape parameters can be seen by the difference in smoothness of the curves in Figure 21 and Figure 22.

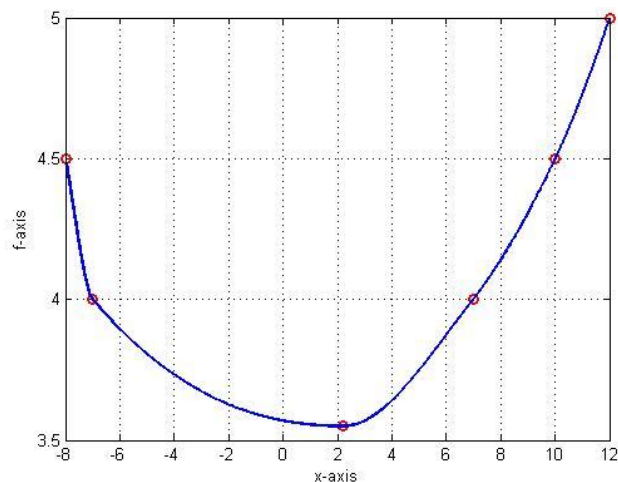


Figure 20. PCHIP curve.

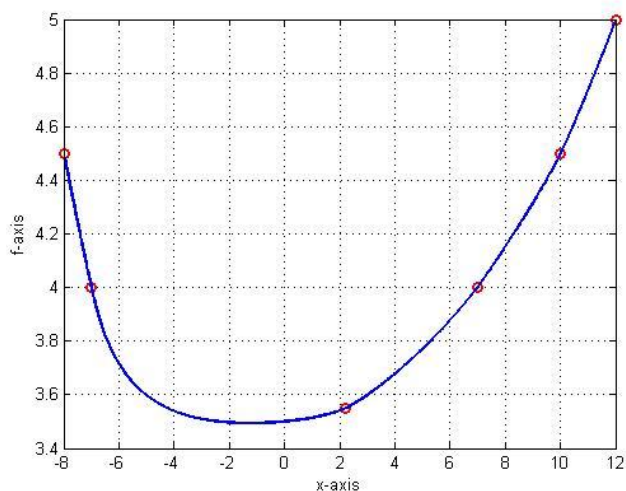


Figure 21. Convexity-preserving curve through convex.

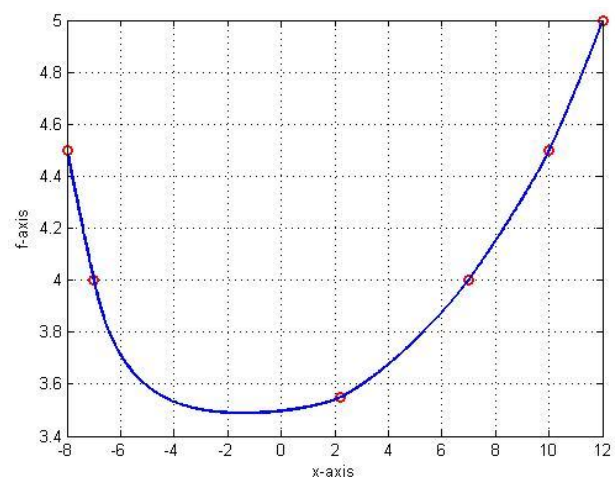


Figure 22. Convexity-preserving curve using C² rational cubic Ball Spline Interpolation.

8. Conclusion

In this work, we have used a C² piecewise rational cubic Ball spline function⁹ to preserve the shape of 2D positive, constrained and convex data in the view of positive, constrained and convex curves. Three shape parameters are used in each curve segment and they are arranged in such a way: one of them is constrained parameters while the remaining two are left free for designer's choice for the modification of positive, constrained and convex curves as desired. The proposed schemes are suitable for such problems in which only data points are given in contrast the schemes^{15,17} imposed a set of constraints on derivatives and data points to obtain the required shape of data. The developed curve schemes have been demonstrated through different numerical examples and observed that the schemes are not only local and computationally economical but are also visually pleasant compared to existing schemes^{3,4,6,7,8,13,14}.

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