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# Generalization of Certain New subclass of P-valent Functions with Negative Coefficients Defined by Differential Operator

**Anas Aljarah and Maslina Darus***

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan, UKM, Bangi - 43600, Selangor DE, Malaysia; anasjrah@yahoo.com; maslina@ukm.edu.my

## Abstract

In this paper we present a new subclass $SM_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ for analytic functions with negative coefficients of the structure $f(z)=z^p-\sum_{n=p+1}^{\infty}a_nz^n$ by using a generalized certain differential operator $S^k_{\alpha,\beta,\lambda,\delta,p}f(z)$ The principle object of this paper is to research the different critical properties and characteristics of the class $SM_{u,p}(\tau,\gamma,\lambda,\alpha,p)$. We additionally determine many results for the Hadamard products of functions belonging to the class $SM_{u,p}(\tau,\gamma,\lambda,\alpha,p)$.

**Keywords:** Analytic Functions, Differential Operator, Hadamard Products, Negative Coefficients, p-Valent Functions

## 1. Introduction and Preliminaries

Let *A(p)* indicate the class of functions of the structure:

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n \quad \left(z\in U,\ p\in\mathbb{N}:=\{1,2,3,\ldots\}\right), \qquad (1.1)$$

which are analytic and $p$-valent in the open unit disc $U=\{z\in\mathbb{C}:|z|<1\}$.

A function *f (z)* in *A(p)* is said to be $p$-valent starlike of order $\gamma$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\gamma \qquad (z\in U), \qquad (1.2)$$

for some $\gamma\,(0\le\gamma<p)$. we indicate by $S^*(\gamma)$ the class of all $p$-valent starlike of order $\gamma$.

Similarly, a function *f (z)* in *A(p)* is said to be $p$-valent convex of order $\gamma$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\gamma \qquad (z\in U), \qquad (1.3)$$

for some $\gamma\,(0\le\gamma<p)$. we indicate by $C(\gamma)$ the class of all $p$-valent convex of order $\gamma$.

**Author for correspondence*

## 2. Definitions and Methodology

For the function $f\in A(p)$, we indicate the following new differential operator:

$$S^0_{\alpha,\beta,\lambda,\delta,p}f(z)=f(z),$$
$$S^1_{\alpha,\beta,\lambda,\delta,p}f(z)=\left[1-p(\lambda-\delta)(\beta-\alpha)\right]f(z)+\left[(\lambda-\delta)(\beta-\alpha)\right]zf'(z),$$

and for $k$=1,2,3,…

$$S^k_{\alpha,\beta,\lambda,\delta,p}f(z)=S\left(S^{k-1}_{\alpha,\beta,\lambda,\delta,p}f(z)\right)$$
$$=z^p+\sum_{n=p+1}^{\infty}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^k a_nz^n, \qquad (1.4)$$

for $f\in A(p),\alpha,\beta,\delta,\lambda\ge0,\lambda>\delta,\beta>\alpha,p\in\mathbb{N}$ and $n\in\mathbb{N}=\mathbb{N}\cup\{0\}$.

It is effectively verified from (1.4) that

$$(\lambda-\delta)(\beta-\alpha)z(S^k_{\alpha,\beta,\lambda,\delta,p}f(z))'$$
$$=S^{k+1}_{\alpha,\beta,\lambda,\delta,p}f(z)-\left(1-p(\lambda-\delta)(\beta-\alpha)\right)S^k_{\alpha,\beta,\lambda,\delta,p}f(z). \qquad (1.5)$$

this is differential operator introduced by[10,11]

***Remark* 1:** (i) When $p = 1$ we have the operator introduced and studied by[8].

With the help of the differential operator $S^k_{\alpha,\beta,\lambda,\delta,p}f(z)$ we define the class $SM_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ as follows:

***Definition* 1:** A function $f(z)\in A(p)^*$ is said to be in the class of $SM_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ if it fulfills the following inequality:

$$\left|\frac{\dfrac{z\tau}{2}\dfrac{(S^k_{\alpha,\beta,\lambda,\delta,p}f(z))'}{\left(S^k_{\alpha,\beta,\lambda,\delta,p}f(z)\right)}+\dfrac{\tau}{2}}{\alpha(p+3\gamma)+\dfrac{2z\alpha\gamma(S^k_{\alpha,\beta,\lambda,\delta,p}f(z))'}{\left(S^k_{\alpha,\beta,\lambda,\delta,p}f(z)\right)}}\right|<1, \qquad (1.6)$$

for some $\begin{matrix}0<\gamma<1, 0<\tau<1, 0<\alpha<1, p\in\mathbb{N}=\{1,2,.....\},\\ \mu,\nu\in\mathbb{R},\lambda\geq 0,\end{matrix}$

and for all $z\in U^*$ we have the class introduced and considered by[1-6], and considered meromorphic univalent and Multivalent functions for diverse classes by[7].

Let $A^*(p)$ denote the subclass of $A(p)$ comprising of functions of the structure:

$$f(z)=z^p-\sum_{n=p+1}^{\infty}a_n z^n \quad \left(z\in U,\ p\in\mathbb{N}:=\{1,2,3,...\}\right), \qquad (1.7)$$

Further, we define the class $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ by

$$S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p) = SM_{u,p}(\tau,\gamma,\lambda,\alpha,p)\cap A^*(p). \qquad (1.8)$$

## 3. Main Results

In this paper we introduce, radii of starlikeness and convexity, closure theorems, extreme points and the class preserving integral operators of the form:

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt \qquad \text{for all } c>-p,$$

for the class $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ is considered. We additionally determine numerous outcomes for the hadamard products of functions belonging to the class $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$.

## 4. Coefficient Inequalities

In this segment, we give an important and adequate condition for a function $f(z)$, given by (1.1), to be in class $SM_{u,p}(\tau,\gamma,\lambda,\alpha,p)$.

***Theorem* 2.1.** *A function $f(z)$ given by (1.1) is in, $SM_{u,p}(\tau,\gamma,\lambda,\alpha,p)$, where $0<\gamma<1, 0<\tau<1, 0<\alpha<1, p\in\mathbb{N}=\{1,2,.....\}$, $\mu,\nu\in\mathbb{R},\lambda\geq 0$*

and $n\in\mathbb{N}_0$ if and only if

$$\sum_{n=p+1}^{\infty}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^k \left[\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right].a_n \leq \frac{\tau}{2}(-p-1)+\alpha\gamma(3+2p)+\alpha p. \qquad (2.1)$$

**Proof**. Assume that $f(z)\in S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$. Then we find from (1.6) that

$$\left|\frac{z\tau}{2}(S^k_{\alpha,\beta,\lambda,\delta,p}f(z))'+\frac{\tau}{2}\left(S^k_{\alpha,\beta,\lambda,\delta,p}f(z)\right)\right| -\left|\alpha(p+3\gamma)\left(S^k_{\alpha,\beta,\lambda,\delta,p}f(z)\right)+2z\alpha\gamma(S^k_{\alpha,\beta,\lambda,\delta,p}f(z))'\right|\leq 0$$

$$\left|\frac{z\tau}{2}(S^k_{\alpha,\beta,\lambda,\delta,p}f(z))'\right|+\left|\frac{\tau}{2}\left(S^k_{\alpha,\beta,\lambda,\delta,p}f(z)\right)\right| -\left|\alpha(p+3\gamma)\left(S^k_{\alpha,\beta,\lambda,\delta,p}f(z)\right)\right|+\left|2z\alpha\gamma(S^k_{\alpha,\beta,\lambda,\delta,p}f(z))'\right|\leq 0$$

$$\left|\frac{z\tau}{2}\left(pz^{p-1}+n\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k a_n z^{n-1}\right)\right| +\left|\frac{\tau}{2}\left(z^p+\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k a_n z^n\right)\right|$$
$$-\left|\alpha(p+3\gamma)\left(z^p+\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k a_n z^n\right)\right| +\left|2z\alpha\gamma\left(pz^{p-1}+n\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k a_n z^{n-1}\right)\right|\leq 0$$

$$z^p\left(\frac{\tau}{2}(-p-1)+\alpha\gamma(3+2p)+\alpha p\right)+ \sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k a_n z^n \left[\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right]\leq 0$$

If we choose $|z|=r<1$, we get

$$\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\left[\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right].a_n$$
$$(0<\gamma<1, 0<\tau<1, 0<\alpha<1, p\in\mathbb{N}),$$
$$\leq\left(\frac{\tau}{2}(-p-1)+\alpha\gamma(3+2p)+\alpha p\right)$$

which is amounting to (2.1).

On the other hand, let us suppose that the equivalent (2.1) remains constant. At that point we have

$$\left|\frac{\dfrac{\dfrac{z\tau}{2}(S^k_{\alpha,\beta,\lambda,\delta,p}f(z))'}{\left(S^k_{\alpha,\beta,\lambda,\delta,p}f(z)\right)}+\dfrac{\tau}{2}}{\alpha\left(p+3\gamma\right)+\dfrac{2z\alpha\gamma(S^k_{\alpha,\beta,\lambda,\delta,p}f(z))'}{\left(S^k_{\alpha,\beta,\lambda,\delta,p}f(z)\right)}}\right|<1,$$

$$\leq\left(\frac{\tau}{2}(-p-1)+\alpha\gamma(3+2p)+\alpha p\right)$$
$$\left(0<\gamma<1, 0<\tau<1, 0<\alpha<1, p\in\mathbb{N}\right),$$

which implies that $f(z)\in S^*M_{u,p}\left(\tau,\gamma,\lambda,\alpha,p\right)$.

This is the required condition. Thus the theorem takes after. □

**Corollary 2.2** *Let the function* $f(z)$ *be defined by* (1.1). *If* $f(z)\in S^*M_{u,p}\left(\tau,\gamma,\lambda,\alpha,p\right)$, *then*

$$a_n\leq \frac{\left(\frac{\tau}{2}(-p-1)+\alpha\gamma(3+2p)+\alpha p\right)}{((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\left[\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right]}.$$
$$(k\geq p;\ p\in\mathbb{N};\ n\in\mathbb{N}_0). \qquad (2.2)$$

*The equality in* (2.2) *is accomplished for the functions* $f(z)$ *given by*

$$f(z)=z^p+\frac{\left(\frac{\tau}{2}(-p-1)+\alpha\gamma(3+2p)+\alpha p\right)}{((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\left[\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right]}z^n$$
$$(k\geq p;\ p\in\mathbb{N};\ n\in\mathbb{N}_0). \qquad (2.3$$

## 5. Radii of Starlikeness and Convexity

In this segment we decide the radii of starlikeness and convexity for the class $S^*M_{u,p}\left(\tau,\gamma,\lambda,\alpha,p\right)$ is given by the accompanying theorems.

**Theorem 3.1.** *Let the function* $f(z)$ *characterized by* (1.7) *is in the class* $S^*M_{u,p}\left(\tau,\gamma,\lambda,\alpha,p\right)$, *where* $0<\gamma<1$, $0<\gamma<1$, $0<\tau<1$, $0<\alpha<1$, $p\in\mathbb{N}$, *and* $n\in\mathbb{N}_0$, *then* $f(z)$ *is starlike of order* $\delta(0\leq\delta<1)$ *in* $|z|<r_1$, *where*

$$r_1=\inf_{n\geq p+1}\left\{\begin{bmatrix}(p-\delta)((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{bmatrix}\div\left[(1-n+p+\delta)\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\right]\right\}^{\frac{1}{n}}$$
$$(n\geq p+1). \qquad (3.1)$$

*The outcome is sharp for the function* $f(z)$ *given by* (2.3).

**Proof**. It suffices to prove that

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq p-\delta, \qquad (3.2)$$

for $|z|<r_2$. We have

$$\left|\frac{zf'(z)}{f(z)}-p\right|=\left|\frac{-\sum_{n=p+1}^{\infty}(1-n)a_nz^n}{z^p+\sum_{n=p+1}^{\infty}a_nz^n}\right|\leq\frac{\sum_{n=p+1}^{\infty}(1-n)a_nz^n}{1+\sum_{n=p+1}^{\infty}a_n|z|^n} \qquad (3.3)$$

Subsequently (3.3) remains constant if

$$\sum_{n=p+1}^{\infty}(1-n)a_n|z|^n\leq(p-\delta)\left(1+\sum_{n=p+1}^{\infty}a_n|z|^n\right)$$

or

$$\sum_{n=p+1}^{\infty}\frac{(1-n-p+\delta)}{(p-\delta)}a_n|z|^n\leq 1 \qquad (3.4)$$

With the guide of (2.1), (3.4) is true if

$$\frac{(1-n+p+\delta)}{(p-\delta)}|z|^n\leq\frac{((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)}{\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)}. \qquad (3.5)$$

Solving (3.5) for $|z|^n$, we obtain

$$r_1=\inf_{n\geq p+1}\left\{\begin{bmatrix}(p-\delta)((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{bmatrix}\div\left[(1-n+p+\delta)\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\right]\right\}^{\frac{1}{n}}$$
$$(n\geq p+1).$$

This is ended the proof of Theorem 3.1. □

**Theorem 3.2.** *Let the function* $f(z)$ *characterized by* (1.7) *is in the class* $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$, *where* $0\le t\le 1$, $0<\tau<1, 0<\alpha<1, p\in\mathbb{N}$, *and* $n\in\mathbb{N}_0$, *then* $f(z)$ *is convex of order* $\delta(0\le\delta<1)$ *in* $|z|<r_2$, *where*

$$r_2=\inf_{n\ge p+1}\left\{\left[\begin{array}{l}p(1-\delta)((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{array}\right]\div\left[n(1-n+p+\delta)\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\right]\right\}^{\frac{1}{n-1}} \quad (3.6)$$

(n ≥ p+1).

*The outcome is sharp for the functions* $f(z)$ *given by* (2.3).

**Proof**. By utilizing the same method employed as a part of the proof of Theorem 4.1, we can demonstrate that

$$\left|\frac{zf''(z)}{f'(z)}\right|\le p-\delta, \quad (3.7)$$

for $|z|<r_2$, with the guide of Theorem 2.1. Accordingly we have the attestation of Theorem 3.2.□

## 6. Closure Theorems

In this segment we prove the closure theorem for the class $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ are given by the following theorems.

Let the functions $f_j(z), j=1,2,\ldots,l$ be characterized by

$$f_j(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,j}z^n\ (a_{n,j}\ge 0), \quad (4.1)$$

for $z\in U$.

**Theorem 5.1.** *Let the functions* $f(z)$ *characterized by* (5.1) *be in the class* $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$, *where* $0<\gamma<1$, $0<\tau<1, 0<\alpha<1, p\in\mathbb{N}$, *and* $n\in\mathbb{N}_0$ *for every* $j=1,2,\ldots,l$. *Then the function G(z) defined by*

$$G(z)=z^p-\sum_{n=p+1}^{\infty}b_nz^n\ (b_n\ge 0) \quad (4.2)$$

*is a member of the class* $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$, *where*

$$b_n=\frac{1}{1}\sum_{j=1}^{1}a_{n,j}\quad (n\ge p+1).$$

*Proof.* Since $f_j(z)\in S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$, it takes after from Theorem 2.1 that

$$\sum_{n=p+1}^{\infty}\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)a_n\end{array}\right]\le\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right).$$

for every $j=1,2,\ldots,l$. Hence,

$$\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)b_n$$

$$=\sum_{n=p+1}^{\infty}\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\times\left\{\frac{1}{l}\sum_{j=1}^{l}a_{n,j}\right\}\end{array}\right]$$

$$=\frac{1}{l}\sum_{j=1}^{l}\left(\begin{array}{l}\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)a_{n,j}\end{array}\right)$$

$$\le\frac{1}{l}\sum_{j=1}^{l}\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)$$

$$=\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right),$$

Which (in perspective of Theorem 2.1) implies that $G(z)\in S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$. □

**Theorem 4.2.** *The class* $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ *is closed under convex linear combination, where* $0<\gamma<1$, $0<\tau<1, 0<\alpha<1, p\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$.

Proof.

$$f_j(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,j}z^n\ (a_{n,j}\ge 0; j=1,2; p\in\mathbb{N}) \quad (4.3)$$

be in the class $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ It is adequate to demonstrate that the function $h(z)$ characterized by

$$h(z)=t f_1(z)+(1-t)f_2(z)\quad (0\le t\le 1) \quad (4.4)$$

is additionally in the class $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$. Since for $0\le t\le 1$,

$$h(z)=z^p-\sum_{n=p+1}^{\infty}\left[ta_{n,1}+(1-t)a_{n,2}\right]z^n\ (0\le t\le 1) \quad (4.5)$$

we watch that

$$\sum_{n=p+1}^{\infty}\begin{bmatrix}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\\ \times\left[ta_{n,1}+(1-t)a_{n,2}\right]\end{bmatrix}$$

$$=t\sum_{n=p+1}^{\infty}\begin{bmatrix}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)a_{n,1}\end{bmatrix}$$

$$+(1-t)\sum_{n=p+1}^{\infty}\begin{bmatrix}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)a_{n,2}\end{bmatrix}$$

$$\le t\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)$$

$$+(1-t)\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)$$

$$=\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right).$$

Hence $h(z)\in S^{*}M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$. This is ended the proof of Theorem 4.2. □

## 7. Integral Operators

In this segment, we consider integral transforms of functions in the class $S^{*}M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$.

**Theorem 5.1.** *If the function $f(z)$ given by (1.7) is in the class $S^{*}M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$, where $0<\gamma<1$, $0<\tau<1, 0<\alpha<1, p\in\mathbb{N}$ and $n\in\mathbb{N}_0$ .and let $c$ be a real number such that $c>-p$. Then the function $F(Z)$ defined by*

$$F(z)=\frac{c+p}{z^{c}}\int_{0}^{z}t^{c-1}f(t)dt \tag{5.1}$$

*likewise belongs to the class $S^{*}M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$.*

*Proof.* From (5.1), it follows that $F(z)=z^{p}-\sum_{n=p+1}^{\infty}b_{n}z^{n}$, where $b_{n}=\left(\frac{c+p}{n+c}\right)a_{n}$. Therefore

$$\sum_{n=p+1}^{\infty}\begin{bmatrix}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)b_{n}\end{bmatrix}$$

$$=\sum_{n=p+1}^{\infty}\begin{bmatrix}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\left(\frac{c+p}{n+c}\right)a_{n}\end{bmatrix}$$

$$\le\sum_{n=p+1}^{\infty}\begin{bmatrix}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)a_{n}\end{bmatrix}\le(1-\alpha p-\beta),$$

since $f(z)\in S^{*}M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$. Hence by Theorem 2.1, $f(z)\in S^{*}M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ □

## 8. Extreme Points

Now, we determine extreme points for the class $S^{*}M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$.

**Theorem 6.1.** *Let* $f_{p}(z)=z^{p}$
and

$$f_{n}=z^{p}-\begin{bmatrix}\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\\ \div\begin{bmatrix}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{bmatrix}\end{bmatrix}z^{n}$$

$$(n\ge p+1;p\in\mathbb{N}),$$

*Then $f(z)\in S^{*}M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ if and only if it can be expressed in the structure*

$$f(z)=\sum_{n=p+1}^{\infty}\xi_{n}f_{n}(z), \tag{6.1}$$

*where $\xi_{n}\ge 0\,(n\ge p)$ and $\sum_{n=p}^{\infty}\xi_{n}=1$.*

*Proof.* Suppose that

$$f(z)=\sum_{n=p}^{\infty}\xi_{n}f_{n}(z)$$

$$=z^{p}-\sum_{n=p+1}^{\infty}\begin{bmatrix}\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\\ \div\begin{bmatrix}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{bmatrix}\end{bmatrix}\xi_{n}z^{n}$$

Then it follows that

$$\sum_{n=p+1}^{\infty}\begin{bmatrix}\begin{bmatrix}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{bmatrix}\\ \div\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\end{bmatrix}$$

$$\cdot\left[\begin{array}{l}\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\\ \div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{array}\right]\end{array}\right]\xi_n\,.$$

$$=\sum_{n=p+1}^{\infty}\xi_n=1-\xi_n\le 1$$

Then from Theorem (2.1) we have $f(z)\in S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$.

Conversely, Let $f(z)\in S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$. Using Corollary 2.2, we have

$$a_n\le\frac{\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)}{((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)}$$

*Setting*

$$\xi_n=\frac{((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)}{\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)}a_n.$$

and $\xi_p=1-\sum_{n=p+1}^{\infty}\xi_n$

We see that $f(z)$ can be expressed in the structure (6.1) this is ended the proof of the theorem (6.1). □

**Corollary 6.2.** *The extreme points of the class* $S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ *are the functions* $f_p(z)=z^p$

$$f_n(z)=z^p-\left[\begin{array}{l}\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\\ \div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{array}\right]\end{array}\right]z^n$$

$(n\ge p+1;\ p\in\mathbb{N})$.

## 9. Convolution Properties

For those are functions

**Theorem 7.1.**

$$f_j(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,j}z^n\left(a_{n,j}\ge 0;\ j=1,2;\ p\in\mathbb{N}\right)\qquad(7.1)$$

we denote by $(f_1\otimes f_2)(z)$ the hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$;that is,

$$(f_1\otimes f_2)(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,1}a_{n,2}z^n.\qquad(7.2)$$

**Theorem 7.2.** *Let the function* $f_j(z)(j=1,2)$ *characterized by* (7.1) *be in the class* $S^*M_{u,p}(\zeta,\gamma,\lambda,\alpha,p)$. *Then* $(f_1\otimes f_2)(z)\in S^*M_{u,p}(\zeta,\gamma,\lambda,\alpha,p)$, *where*

$$\zeta=\frac{2\alpha\gamma(2p+3)-2\alpha p}{(p-1)}+\left[\begin{array}{l}2\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)^2\\ \div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)+1)^k\\ \times\left(\frac{\tau}{2}(p+2)-\alpha\gamma(3-2(p+1))-\alpha p\right)(p-1)\end{array}\right]\end{array}\right].\qquad(7.3)$$

*The outcome is sharp for the functions* $f_j(z)(j=1,2)$ *given by*

$$f_j(z)=z^p-\left[\begin{array}{l}2\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)^2\\ \div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)+1)^k\\ \times\left(\frac{\tau}{2}(p+2)-\alpha\gamma(3-2(p+1))-\alpha p\right)(p-1)\end{array}\right]\end{array}\right]z^n\qquad(7.4)$$

**Proof**. Employing the procedure utilized before by[9], we require to discover. the biggest $\zeta$ such that

$$\sum_{n=p+1}^{\infty}\left[\begin{array}{l}\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{array}\right]\\ \div\left(\frac{\zeta}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\end{array}\right]a_{n,1}a_{n,2}\le 1\qquad(7.5)$$

For $f_j(z)\in S^*M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$ $(j=1,2)$ since $f_j(z)\in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$ $(j=1,2)$, we readily see that

$$\sum_{n=p+1}^{\infty}\left[\begin{array}{l}\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{array}\right]\\ \div\left[\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\right]\end{array}\right]a_{n,j}\le 1\qquad(7.6)$$

$(j=1,2)$.

In this way, by the Cauchy-Schwarz inequality, we get

$$\sum_{n=p+1}^{\infty}\left[\begin{array}{l}\left[\begin{array}{l}\left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{array}\right]\\ \div\left[\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\right]\end{array}\right]\sqrt{a_{n,1}a_{n,2}}\leq 1. \quad (7.7)$$

This implies that we need only to show that

$$\frac{a_{n,1}a_{n,2}}{\left(\frac{\zeta}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)}\leq\frac{\sqrt{a_{n,1}a_{n,2}}}{\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)}(n\geq p+1) \quad (7.8)$$

or, equivalently, that

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{\left(\frac{\zeta}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)}{\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)} \quad (n\geq p+1). \quad (7.9)$$

Consequently, by the inequality (7.7), it is adequate to prove that

$$\left[\begin{array}{l}\left[\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\right]\\ \div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)\end{array}\right]\end{array}\right]\leq\frac{\left(\frac{\zeta}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)}{\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)}(n\geq p+1). \quad (7.10)$$

It follows from (7.10) that

$$\zeta\leq\frac{2\alpha\gamma(2p+3)-2\alpha p}{(p-1)}+\left[\begin{array}{l}\left[2\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)^{2}\right]\\ \div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)(p-1)\end{array}\right]\end{array}\right](n\geq p+1). \quad (7.11)$$

Now, defining the function $\varphi(z)$ by

$$\varphi(n)=\frac{2\alpha\gamma(2p+3)-2\alpha p}{(p-1)}+\left[\begin{array}{l}\left[2\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)^{2}\right]\\ \div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^{k}\\ \times\left(\frac{\tau}{2}(n+1)-\alpha\gamma(3-2n)-\alpha p\right)(p-1)\end{array}\right]\end{array}\right](n\geq p+1), \quad (7.12)$$

we see that $\varphi(z)$is an expanding function of n. In this way, we infer that

$$\zeta\leq\varphi(p)=\frac{2\alpha\gamma(2p+3)-2\alpha p}{(p-1)}+\left[\begin{array}{l}\left[2\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)^{2}\right]\\ \div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)+1)^{k}\\ \times\left(\frac{\tau}{2}(p+2)-\alpha\gamma(3-2(p+1))-\alpha p\right)(p-1)\end{array}\right]\end{array}\right], \quad (7.13)$$

which apparently completes the proof of Theorem (7.2). □

Utilizing contentions like those as a part of the proof of Theorem (7.2), we acquire the taking after result.

**Theorem 7.3.** *Let the function $f(z)$ characterized by (7.1) be in the class $S^{*}M_{u,p}(\zeta,\gamma,\lambda,\alpha,p)$. assume also that the function $f_2(z)$ characterized by (7.1) be in the class $S^{*}M_{u,p}(\tau,\gamma,\lambda,\alpha,p)$*

*Then $(f_1\otimes f_2)(z)\in S^{*}M_{u,p}(\omega,\gamma,\lambda,\alpha,p)$, where*

$$\omega=\frac{2\alpha\gamma(2p+3)-2\alpha p}{(p-1)}+\left[\begin{array}{l}\left[\begin{array}{l}2\left(\frac{\tau}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\\ \times\left(\frac{\zeta}{2}(-1-p)-\alpha\gamma(3+2p)+\alpha p\right)\end{array}\right]\\ \div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)+1)^{k}\\ \times\left(\frac{\tau}{2}(p+2)-\alpha\gamma(3-2(p+1))-\alpha p\right)(p-1)\end{array}\right]\end{array}\right], \quad (7.14)$$

The outcome is sharp for the functions $f_j(z)(j=1,2)$ given by.

$$f_1(z) = z^p - \left[\frac{\left[2\left(\frac{\tau}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)\right]}{\div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)+1)^k \\ \times\left(\frac{\tau}{2}(p+2) - \alpha\gamma(3-2(p+1)) - \alpha p\right)(p-1)\end{array}\right]}\right] z^n \quad (7.15)$$

$(p \in \mathbb{N}\, k \in \mathbb{N}_0)$

and

$$f_2(z) = z^p - \left[\frac{\left[2\left(\frac{\zeta}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)\right]}{\div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)+1)^k \\ \times\left(\frac{\tau}{2}(p+2) - \alpha\gamma(3-2(p+1)) - \alpha p\right)(p-1)\end{array}\right]}\right] z^n \quad (7.16)$$

$(p \in \mathbb{N}\, k \in \mathbb{N}_0)$.

**Theorem 7.4.** *Let the function* $f_j(z)(j=1,2)$ *characterized by* (7.1) *be in the class* $S^* M_{u,p}(\omega, \gamma, \lambda, \alpha, p)$.

*Then* $h(z)$ *defined by*

$$h(z) = z^p - \sum_{n=p+1}^{\infty} \left(a_{n,1}^2 + a_{n,2}^2\right) z^n \quad (7.17)$$

*belongs to the class* $S^* M_{u,p}(\omega, \gamma, \lambda, \alpha, p)$ *where*

$$\omega = \frac{2\alpha\gamma(2p+3) - 2\alpha p}{(p-1)} + \left[\frac{\left[2\left(2\left(\frac{\tau}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)^2\right)\right]}{\div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)+1)^k \\ \times\left(\frac{\tau}{2}(p+2) - \alpha\gamma(3-2(p+1)) - \alpha p\right)(p-1)\end{array}\right]}\right]. \quad (7.18)$$

The *outcome* is sharp for the functions $f_j(z)(j=1,2)$ *given already by* (7.4).

**Proof**. Refer to

$$\sum_{n=p+1}^{\infty} \left[\frac{\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k \\ \times\left(\frac{\tau}{2}(n+1) - \alpha\gamma(3-2n) - \alpha p\right)\end{array}\right]^2}{\div\left[\left(\frac{\tau}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)^2\right]}\right] a_{n,j}^2$$

$$\le \left(\sum_{n=p+1}^{\infty} \left[\frac{\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k \\ \times\left(\frac{\tau}{2}(n+1) - \alpha\gamma(3-2n) - \alpha p\right)\end{array}\right]\div}{\left(\frac{\tau}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)}\right] a_{n,j}\right)^2 \le 1 \quad (7.19)$$

$(j=1,2)$,

For $f_j(z) \in S^* M_{u,p}(\tau, \gamma, \lambda, \alpha, p)(j=1,2)$, we have

$$\sum_{n=p+1}^{\infty} \frac{1}{2} \left[\frac{\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k \\ \times\left(\frac{\tau}{2}(n+1) - \alpha\gamma(3-2n) - \alpha p\right)\end{array}\right]^2}{\div\left(\frac{\tau}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)^2}\right] \left(a_{n,1}^2 + a_{n,2}^2\right) \quad (7.20)$$

$\le 1$.

Subsequently, we need to .locate the biggest $\omega$ such that

$$\frac{1}{\left(\frac{\omega}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)} \le \left[\frac{\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k \\ \times\left(\frac{\tau}{2}(n+1) - \alpha\gamma(3-2n) - \alpha p\right)\end{array}\right]}{\div\left[2\left(\frac{\tau}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)^2\right]}\right] (n \ge p+1), \quad (7.21)$$

that is, that

$$\omega = \frac{2\alpha\gamma(2p+3) - 2\alpha p}{(p-1)} + \left[\frac{2\left(2\left(\frac{\tau}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)^2\right)}{\div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k \\ \times\left(\frac{\tau}{2}(n+1) - \alpha\gamma(3-2n) - \alpha p\right)(p-1)\end{array}\right]}\right] (n \ge p+1), \quad (7.22)$$

Now, defining a function $\psi(n)$ by

$$\psi(n) = \frac{2\alpha\gamma(2p+3) - 2\alpha p}{(p-1)} + \left[\frac{2\left(2\left(\frac{\tau}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p\right)^2\right)}{\div\left[\begin{array}{l}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k \\ \times\left(\frac{\tau}{2}(n+1) - \alpha\gamma(3-2n) - \alpha p\right)(p-1)\end{array}\right]}\right] \quad (7.23)$$

$(n \ge p+1)$.

we observe that $\psi(n)$ is an expanding function of n. We reason that

$$\omega = \frac{2\alpha\gamma(2p+3)-2\alpha p}{(p-1)} + \left[ \begin{array}{l} \left[ 2\left( 2\left( \frac{\tau}{2}(-1-p) - \alpha\gamma(3+2p) + \alpha p \right)^2 \right) \right] \\ \div \left[ \begin{array}{l} \left( (\lambda-\delta)(\beta-\alpha)+1 \right)^k \\ \times \left( \frac{\tau}{2}(p+2) - \alpha\gamma(3-2(p+1)) - \alpha p \right)(p-1) \end{array} \right] \end{array} \right]. \quad (7.24)$$

which is ended the proof of Theorem (7.4). □

## 10. Acknowledgement

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## 11. References

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