Indian Journal of Science and Technology, *Vol 8(34), DOI: 10.17485/ijst/2015/v8i34/50621, December 2015*
ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645

Dynamical Control of Accuracy using the Stochastic Arithmetic to Estimate the Solution of the Fuzzy Differential Equations via Homotopy Analysis Method

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Abstract

One of the important analytical-approximate schemes inorder to obtain the solution of a differential equation is the Homotopy Analysis Method (HAM). In this research, a novel technique is considered to validate the results of the algorithm obtained from the HAM to find the solution of a fuzzy differential equation with initial condition based on the generalized differentiablity. For this purpose, in place of the current floating-point arithmetic, a new arithmetic which is called the stochastic arithmetic is replaced. To this aim, the CESTAC (Controle et Estimation Stochastique des Arrondis de Calculs)method is applied which replaces the floating-point arithmetic by the stochastic arithmetic. Also, a numerical algorithm is presented to determine the steps of using the CESTAC method to find the numerical solution of a fuzzy differential equation at a given point by means of the HAM. In order to determine the accuracy of the proposed method, a theorem is proved. By using the proposed scheme, the optimal number of steps and the optimal auxiliary parameter in the HAM can be found and the results are computed in a valid way with their accuracy. Also, the stability of the method is verified and the results will be determined with their correct significant digits. Finally, two sample fuzzy differential equations are solved based on the mentioned algorithm to illustrate the importance, advantages and applicability of using the stochastic arithmetic in place of the floating-point arithmetic. The programs have been provided by MAPLE package.

Keywords:

1. Introduction

a lot of works have been done to solve fuzzy differential equations via different,In recent years

numerical and analytical methods. The algorithms of these works have been implemented by mathematical packages based on the floating-point arithmetic. From the point of view of numerical computations, the validation of final solution at a given point is important. Since the computer arithmetic is not able to validate the results, we must substitute other arithmetic,which is the stochastic arithmetic, in order to rely the steps of algorithm and also the final results.

The Homotopy Analysis Method (HAM) proposed by Liao[27-29] is a well known semi-analytical method to solve the linear and nonlinear differential equations and has been widely applied to solve different problems such as[11,12,24-26]. In this method, the solution is obtained as a seriesform based on a concept of deformation in homotopy and a recursive relation. In this work, the CESTAC method is applied which is based on the stochastic arithmetic[3-7,21,23] in order to find the accuracy

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of the HAM andfind the optimal solution of the ordinary fuzzy differential equation. To this end, the number of the significant digits of the computed results which are common with the exact solution are calculated. The CESTAC method which was developed by La Porte and Vignes[32] is based on aprobabilistic approach of the round-off error propagation which replaces the floating-point arithmetic by a random arithmetic[20]. By using this method, N runs of the computer program take place in parallel. In this case, a new arithmetic called stochastic arithmetic is defined.

The definitions and properties of the stochastic arithmetic have been explained in[20,21,32]. Also, some of the applications of this arithmetic have been presented in[3-7,23]. The basic idea of the CESTAC method[17-19] is to replace the usualfloating-point arithmetic with a randomarithmetic. Consequently, each result appearsas a random variable.

This method is able to detect numerical instabilities which occur during the run and estimate accuracy of the computed results. During the run, as soon as the number of the significant digits of any result becomes zero, an informatical zero is detected and theresult is printed by the notation @0.

Let F be the set of all machine numbers. Thus, any real value r is represented in the form of $R \in$ F in the computerIn the binary floating-point arithmetic with P mantissa bits, the rounding error stems from assignment operator is,

$$R = r - \varepsilon 2^{E-P} \alpha, \tag{1}$$

where, ε is the sign of r and 2^{-P} α is the lost part of the mantissa due to round-off error and E is the binary exponent of the result. Also, in single precision, $P = 24$. and in double precision, $P = 54$. According to (1), in order to perturb the last mantissa bit of the value, r it is sufficient that the value α is considered as a random variable uniformly distributed on [-1,1] (perturbation method). Thus the computed result, R is a random variable and its precision depends on its mean and its standard deviation respectively denoted by μ and σ.

If a computer program is performed times, the distribution of the results R_i, $i = 1,..., n$, is quasi-Gaussian which their mean is equal to the exact real value r, that is $E(\overline{R}) = r$[19,32]. This N samples are used for estimating the values μ and σ. The samples R_i are obtained by perturbation of the last mantissa bit or previous bits (if necessary) of every result R, then the mean of random samples R_i, is considered as the result of an arithmetic operation. In CESTAC method if $C_{\overline{R}r} < 0,$ the informatical result R is insignificant and it means a numerical instability exists in its related line. The value n can be chosen any natural number but in order to avoid the execution time, usually $n = 3$. In this case, the number of the exact significant digits common to R and to the exact value r can be estimated by[19],

$$C_{\overline{R},r} = \log_{10} \frac{|\overline{R}|}{S}, \tag{2}$$

where S is the standard deviation of the samples R_i. If $C_{\overline{R}r} < 0,$ there is an instability in the evaluated result.

In section 2, some preliminaries of fuzzy mathematics are mentioned. In section 3, the HAM is explained to solve a fuzzy differential equation with initial conditions. The main idea is proposed in section 4 in which a theorem is proved to illustrate the accuracy of the HAM in solving a fuzzy differential equation. Two numerical examples are solved based on the given algorithm in section 5 to verify the validity and importance of using the CESTAC method based on the stochastic arithmetic.

2. Preliminaries

We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)),\ 0 \le r \le 1,$ which satisfy the following conditions[13,22,30,33]

- $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1].
- $\overline{u}(r)$ is a bounded left continuous non-increasing function over [0,1].
- $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1.$

The set of all fuzzy numbers is denoted by F(ℝ).Let two fuzzy numbers u, v and real number k then,

- $\underline{(u+v)}(r) = \underline{u}(r) + \underline{v}(r)$
- $\overline{(u+v)}(r) = \overline{u}(r) + \overline{v}(r)$
- $\underline{(ku)}(r) = k\underline{u}(r), \quad \overline{(ku)}(r) = k\overline{u}(r), \quad k \ge 0$
- $\underline{(ku)}(r) = k\overline{u}(r), \quad \overline{(ku)}(r) = k\underline{u}(r), \quad k \le 0$
- $\underline{(u.v)}(r) = \min\{\underline{u}(r)\underline{v}(r), \underline{u}(r), \overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$
- $\overline{(u.v)}(r) = \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r), \overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$

Definition 2.1. For arbitrary fuzzy numbers $\tilde{u} = (\underline{u}(r), \overline{u}(r)), \tilde{v} = (\underline{v}(r), \overline{v}(r))$ the quantity

$$D(\tilde{u}, \tilde{v}) = \sup_{0 \le r \le 1}\{\max[|\underline{u}(r)\underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|]\}$$

is the Hausdorff distance between and $\tilde{u}$ and $\tilde{v}$.

Definition 2.2. A function f:ℝ→F(ℝ) is called a fuzzy function. If for arbitrary fixed $t_0 \in F(\mathbb{R})$ and ε > 0 such that, $|t-t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \varepsilon$ exists, *f* is said to be continuous.

Definition 2.3. Let. $\tilde{u},\tilde{v} \in F(\mathbb{R})$. If there exists $\tilde{w} \in F(\mathbb{R})$ such that $\tilde{u} = \tilde{v} + \tilde{w}$, then $\tilde{w}$, is called the H-difference of $\tilde{u},\tilde{v}$ and it is denoted $\tilde{u} \ominus \tilde{v}$.

Definition 2.4. Let *a*, $b \in \mathbb{R}$ and *f*: $(a,b) \to F(\mathbb{R})$. *f* is said to be strongly generalized differentiable at the fixed point $t_0 \in (a,b)$ if there exists $f'(t_0) \in F(\mathbb{R})$ such that

- for all $h > 0$ sufficiently close to 0, ∃ $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus$, $(t_0 - h)$ and the limits

$$\lim_{h\to 0^+} \frac{f(t_0+h) \odot f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0) \odot f(t_0-h)}{h} = f'(t_0)$$

or

- for all $h > 0$ sufficiently close to 0, ∃ $f(t_0 - h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 + h)$, and the limits

$$\lim_{h\to 0^+} \frac{f(t_0-h) \odot f(t_0)}{-h} = \lim_{h\to 0^+} \frac{f(t_0) \odot f(t_0+h)}{-h} = f'(t_0)$$

or

- for all $h > 0$ sufficiently close to 0, ∃ $f(t_0 + h) \ominus f(t_0)$, $f(t_0 - h) \ominus f(t_0)$, and the limits

$$\lim_{h\to 0^+} \frac{f(t_0+h) \odot f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0-h) \odot f(t_0)}{h} = f'(t_0)$$

or

- for all $h > 0$ sufficiently close to 0, ∃ $f(t_0) \ominus f(t_0 + h)$, $f(t_0) \ominus f(t_0 - h)$, and the limits

$$\lim_{h\to 0^+} \frac{f(t_0) \odot f(t_0+h)}{-h} = \lim_{h\to 0^+} \frac{f(t_0) \odot f(t_0-h)}{h} = f'(t_0)$$

(*h* and - *h* at denominators mean and $\frac{1}{h}$ and $\frac{-1}{h}$ respectively).

Theorem 2.1. Let *f*: $(a,b) \to F(\mathbb{R})$ be strongly generalized differentiable on each point $t \in (a,b)$ in sense of definition 2.4, (iii), (iv). Then $f'(t) \in \mathbb{R}$ for all $t \in (a,b)$.

Theorem 2.2. Let $f : \mathbb{R} \to F(\mathbb{R})$ be a function and denote $f(t) = (\underline{f}(t,r), \overline{f}(t,r))$, for each $r \in [0.1]$. Then,

- If *f* is differentiable in the first form (i), then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable and $f'(t) = (\underline{f}'(t,r), \overline{f}'(t,r))$,
- If *f* is differentiable in the first form (ii), then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable and $f'(t) = (\overline{f}'(t,r), \underline{f}'(t,r))$

3. HAM for Fuzzy Differential Equation

In this section, the HAMis introduced for the following fuzzy differential equations[1,2,8-10,14-16]:

$$N(\tilde{u}(t)) = \tilde{c} \quad (3)$$

where *N* is a nonlinear operator, *t* denote the independent variables, $\tilde{u}$ is an unknown fuzzy function and $\tilde{c}$ is a known fuzzy function.

At first, we consider $\tilde{u}(t)=(\underline{u}(t,r),\overline{u}(t,r)),\tilde{c}(t)=(\underline{c}(t,r),\overline{c}(t,r))$. According to the type of differentiability of , we can rewrite the Eq.(3) in the following form:

$$\begin{cases} N_1(\underline{u},\overline{u},\underline{c}) = 0, \\ N_2(\underline{u},\overline{u},\overline{c}) = 0 \end{cases} \quad (4)$$

By means of the HAM, we construct the zeroth-order deformation equation

$$\begin{cases} L[\phi_1(t;q) - \underline{u}_0(t)] = qh_1H_1(t)N_1[\phi_1(t;q),\phi_2(t;q),\underline{c}(t,r)] \\ L[\phi_2(t;q) - \overline{u}_0(t)] = qh_2H_2(t)N_2[\phi_1(t;q),\phi_2(t;q),\overline{c}(t,r)] \end{cases} \quad (5)$$

where $q \in [0,1]$ is the embedding parameter, h_1, h_2 are auxiliary parameters, *L* is an auxiliary linear operator $H_1(t)$, $H_2(t)$ and are auxiliary functions. $\phi_1(t;q)$, $\phi_2(t;q)$ are unknown functions and $\underline{u}_0(t),\overline{u}_0(t)$ are initial guesses of $\underline{u}(t),\overline{u}(t)$ respectively. It is obvious that when $q = 0$ and $q = 1$, we have:

$$\phi_1(t;0) = \underline{u}_0(t), \quad \phi_1(t;1) = \underline{u}(t),$$
$$\phi_2(t;0) = \overline{u}_0(t), \quad \phi_2(t;1) = \overline{u}(t),$$

Therefore, as *q* increases from 0 to 1, the solutions $\phi_1(t;q)$, $\phi_2(t;q)$ varies from the $\underline{u}_0(t),\overline{u}_0(t)$ to the exact solution $\underline{u}(t),\overline{u}(t)$ respectively. By Taylor's theorem, we expand $\phi_1(t;q)$, $\phi_2(t;q)$ in a power series of the embedding parameter *q* as follows:

$$\phi_1(t;q) = \underline{u}_0(t), \; + \sum_{m=1}^{+\infty} \underline{u}_m(t) q^m,$$
$$\phi_2(t;q) = \overline{u}_0(t), \; + \sum_{m=1}^{+\infty} \overline{u}_m(t) q^m \quad (6)$$

Where

$$\underline{u}_m(t) = \frac{1}{m!} \frac{\partial^m \phi_1(t:q)}{\partial q^m} |q = 0,$$
$$\overline{u}_m(t) = \frac{1}{m!} \frac{\partial^m \phi_2(t:q)}{\partial q^m} |q = 0 \quad (7)$$

Let the initial guesses $\underline{u}_0(t),\overline{u}_0(t)$, the auxiliary linear operator *L*, the nonzero auxiliary parameters h_1, h_2 and the auxiliary functions $H_1(t)$, $H_2(t)$ be properly chosen so that the power series (6) converges at $q = 1$, then, we have:

$$\overline{u}(t) = \underline{u}_0(t), + \sum_{m=1}^{+\infty} \underline{u}_m(t),$$
$$\overline{u}(t) = \overline{u}_0(t), + \sum_{m=1}^{+\infty} \overline{u}_m(t) \quad (8)$$

Which must be the solution of the original nonlinear equations. Now, we define the following sets of vectors:

$$\vec{\underline{u}}_n = \{\underline{u}_0(t), \underline{u}_1(t), \ldots, \underline{u}_n(t)\},$$
$$\vec{\overline{u}}_n = \{\overline{u}_0(t), \overline{u}_0(t), \ldots, \overline{u}_n(t)\} \tag{9}$$

By differentiating the zeroth order deformation equations (5) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing by $m!$, we get the following m-th order deformation equations:

$$L[\underline{u}_m(t) - \chi_m \underline{u}_{m-1}(t)] = \underline{u}_m(t) R_{1_m}(\vec{\underline{u}}_{m-1}, \vec{\overline{u}}_{m-1}),$$
$$L[\overline{u}_m(t) - \chi_m \overline{u}_{m-1}(t)] = \overline{u}_m(t) R_{2_m}(\vec{\underline{u}}_{m-1}, \vec{\overline{u}}_{m-1}) \tag{10}$$

Where,

$$R_{1_m}\left(\vec{\underline{u}}_{m-1}, \vec{\overline{u}}_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_1[\phi_1(t;q),]\phi_2(t;q), \underline{c}(t,r)}{\partial q^{m-1}} | q = 0,$$
$$R_{2_m}\left(\vec{\underline{u}}_{m-1}, \vec{\overline{u}}_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_2[\phi_1(t;q),]\phi_2(t;q), \overline{c}(t,r)}{\partial q^{m-1}} | q = 0 \tag{11}$$

And

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{12}$$

It should be emphasized that $\underline{u}_m(t), \overline{u}_m(t) \, for \, m \ge 1$ is governed by the linear equations (10) with boundaryconditions that come from the original problem. For more details about HAM, we refer the reader to[11,12,24-29].

4. Main Idea

In this section, we use the stochastic arithmetic in order to estimate $\underline{u}, \overline{u}$ in Equation(4) by using the HAM in various value of t. At first, we introduce the following definition which has been mentioned in[3].

Definition 4.1 Let r and r^1 be two real numbers, the number of significant digits that are common to a and b, denoted C $_{r,r'}$ by can be defined by

$$If \; r \ne r' \; then \; C_r, C_{r'} = \log_{10}\left|\frac{r + r'}{2(r - r')}\right|, otherwise \; C_{r,r} = +\infty$$

Now, the following theorem about the accuracy of the HAM to estimate $\underline{u}, \overline{u}$ is proved.

Theorem 4.1 *Let* $\underline{v}_m(t) = \sum_{i=0}^{m} \underline{u}_i, \overline{v}_m = \sum_{i=0}^{m} \overline{u}_i$ be approximate value of $\underline{u}(t), \overline{u}(t)$ in Equation (8)

for any t obtained from the HAM, then

$$C\underline{v}_m, \underline{v}_{m-1} = C\underline{v}_m, u + 0\left(\frac{1}{m}\right), \tag{13}$$

$$C\overline{v}_m, \overline{v}_{m-1} = C\overline{v}_m, u + 0\left(\frac{1}{m}\right) \tag{14}$$

Proof. In order to prove Equation13, according to the definition 4.1, we get :

$$C\underline{v}_m, \underline{v}_{m-1} - C\underline{v}_m = \log_{10}\left|\frac{\underline{v}_m + \underline{v}_{m+1}}{2(\underline{v}_m - \underline{v}_{m+1})}\right| - \log_{10}\left|\frac{\underline{v}_m + \underline{u}}{2(\underline{v}_m - \underline{u})}\right| =$$
$$\log_{10}\left|\frac{\underline{v}_m + \underline{v}_{m+1}}{2(\underline{u}_{m+1})}\right| - \log_{10}\left|\frac{\underline{v}_m + \underline{u}}{2(\underline{v}_m - \underline{u})}\right| = \log_{10}\left|\frac{\underline{v}_m + \underline{v}_{m+1}}{2(\underline{v}_m - u)}\right| - \log_{10}\left|\frac{\underline{u}_{m+1}}{(\underline{v}_m - \underline{u})}\right| =$$
$$\log_{10}\left|\frac{\underline{v}_m + \underline{v}_{m+1}}{(\underline{v}_m + u)}\right| + \log_{10}\left|\frac{\underline{v}_m + \underline{u}}{\underline{u}_{m+1}}\right| \tag{15}$$

The first term of Equation (15) vanishes, since $\underline{v}_m$ *tend to* $\underline{u}$ when m increases. The second term of Equation(15) can be written as,

$$\log_{10}\left|\frac{\underline{v}_m + \underline{u}}{\underline{u}_{m+1}}\right| - \log_{10}\left|\frac{\sum_{i=0}^{m}\underline{u}_i - \sum_{i=0}^{+\infty}\underline{u}_i}{\underline{u}_{m+1}}\right| = \log_{10}\left|\frac{\sum_{i=m+1}^{+\infty}\underline{u}_i}{\underline{u}_{m+1}}\right| = \log_{10}\left|\frac{\sum_{i=m+2}^{+\infty}\underline{u}_i}{\underline{u}_{m+1}}\right| =$$
$$\log_{10}\left|1 + \frac{\sum_{i=m+2}^{+\infty} h_1 L^{-1}\left[H_1(t) R_{1_i}(\vec{\underline{u}}_{i-1}, \vec{\overline{u}}_{i-1})\right]}{h_1 L^{-1}\left[H_1(t) R_{1_{m+1}}(\vec{\underline{u}}_m, \vec{\overline{u}}_m)\right]}\right|$$

According to Equation (11), we get:

$$\frac{\sum_{i=m+2}^{+\infty} h_1 L^{-1}\left[H_1(t) R_{1_i}(\vec{\underline{u}}_{i-1}, \vec{\overline{u}}_{i-1})\right]}{h_1 L^{-1}\left[H_1(t) R_{1_{m+1}}(\vec{\underline{u}}_m, \vec{\overline{u}}_m)\right]} =$$
$$\frac{\sum_{i=m+2}^{+\infty} h_1 L^{-1}\left[H_1(t) \frac{1}{(i-1)!} \frac{\partial^{i-1} N_1[\phi_1(t;q), \phi_2(t;q), \underline{c}(t,r)]}{\partial q^{i-1}} | q = 0\right]}{h_1 L^{-1}\left[H_1(t) \frac{1}{m!} \frac{\partial^m N_1[\phi_1(t;q), \phi_2(t;q), \underline{c}(t,r)]}{\partial q^m} | q = 0)\right]} =$$
$$\frac{h_1 \frac{1}{(m+1)!} L^{-1}\left[H_1(t) \frac{\partial^{m+1} N_1[\phi_1(t;q), \phi_2(t;q), \underline{c}(t,r)]}{\partial q^{i-1}} | q = 0\right]}{h_1 \frac{1}{m!} L^{-1}\left[H_1(t) \frac{\partial^m N_1[\phi_1(t;q), \phi_2(t;q), \underline{c}(t,r)]}{\partial q^m} | q = 0)\right]} +$$
$$\frac{h_1 \frac{1}{(m+2)!} L^{-1}\left[H_1(t) \frac{\partial^{m+2} N_1[\phi_1(t;q), \phi_2(t;q), \underline{c}(t,r)]}{\partial q^{i-1}} | q = 0\right]}{h_1 \frac{1}{m!} L^{-1}\left[H_1(t) \frac{\partial^m N_1[\phi_1(t;q), \phi_2(t;q), \underline{c}(t,r)]}{\partial q^m} | q = 0)\right]} + \ldots$$

Therefore, $\log_{10}\left|\frac{\underline{v}_m - \underline{u}}{\underline{u}_{m+1}}\right| = 0\left(\frac{1}{m}\right)$, Equation(13) is proved. Similarly, Equation(14) can be proved.

The relations (13) and (14) show that the number of significant digits in common between $\underline{v}_m, \underline{v}_{m-1}$ *and* $\overline{v}_m, \overline{v}_{m-1}$, are in common with the exact value $\underline{u}$ *and* $\overline{u}$ respectively. Also, for m large enough $0\left(\frac{1}{m}\right) << 1$.

Based on the theorem 4.1, when the difference between two sequential results is informatical zero, the optimal number of iterations is achieved and the next iterations are redundant. The following algorithm evaluates $\underline{u}$ *and* $\overline{u}$ at the point t in the stochastic arithmetic and by applying the CESTAC method:

Algorithm

- Read $\underline{u}_0, \overline{u}_0, h_1, h_2, H_1, H_2, t$.
- Set $\underline{v}_0 = \underline{u}_0, \overline{v}_0 = \overline{u}_0$ *and* $m = 1$.
- Evaluate $\underline{v}_1, \overline{v}_1$ by using the HAM in the stochastic arithmetic.
- While $\left(|\underline{v}_m(t) - \underline{v}_{m-1}(t)| \neq @0 \& |\overline{v}_m(t) - \overline{v}_{m-1}(t)| \neq @0\right)$ Do

  4.a. Set $m = m + 1$.

  4.b. Evaluate $\underline{v}_m, \overline{v}_m$ by using the HAM in the stochastic arithmetic.
- Print $m, \underline{v}_m, \overline{v}_m$.

5. Numerical Examples

In this section, two sample examples are solved based on the proposed algorithm. In order to validate the results and find the optimal number of iterations at a given points, we apply the CESTAC method. The programs have been provided by MAPLE in the stochastic arithmetic.

Example 5.1

We consider the following fuzzy differential equation

$$\begin{cases} \tilde{u}' - \tilde{u} = \tilde{0}, \\ \tilde{u}(0) = \tilde{1} \end{cases} \tag{16}$$

Where, $\tilde{0} = ((3r-3)e^t, (3-3r)e^t)$ *and* $\tilde{1} = (2r-1, 2-r)$ .The exact solution of this equation is ((2*r* -1) e^t, (2-*r*)e^t), then we can conclude $\tilde{u}$ is differentiable in the form (i), therefore for applying HAM, we have

$$\begin{cases} (2r-1)\underline{u}' - (2-r)\overline{u} = (3r-3)e^t, \\ (2r-1)\overline{u}' - (2r-1)\overline{u} = (3-3r)e^t, \\ \underline{u}(0) = 3r-3, \\ \overline{u}(0) = 3-3r \end{cases} \tag{17}$$

The results of applying the HAM by cestac method on the Equation (17) by selection h_1, h_2 = - 1and H_1, H_2 = 1 at *t* = 2 in the levels $r = \frac{1}{3}, \frac{2}{3}$ are gathered in the Tables 1and 2 respectively.

Table 1 shows that the optimal number of terms in the HAM for $\underline{v}_m$ is m=14 with optimal value -2.4630186 and for $\overline{v}_m$ is m=15 with optimal value 12.31509399 when $r = \frac{1}{3}$. Similarly, Table 2 shows that the optimal numberof terms in the HAM for $\underline{v}_m$ is m=16 with optimal value 2.463018797 and for $\overline{v}_m$ is m=16 with optimal value 9.852075187 when $r = \frac{2}{3}$.

Example 5.2

Let the following fuzzy Newton's equation

$$\begin{cases} \tilde{1}\tilde{u}' + \tilde{u}^2 = \tilde{1}' \\ \tilde{u}(0) = 0 \end{cases} \tag{18}$$

Where, $\tilde{1} = (r, 2-r)$ *and* $\tilde{1}' = (r^2, (2-r)^2)$ .The exact solution of Equation (18) is (*r* tanh *t*, (2-*r*)tanh *t*), then we can conclude ũ is differentiable in the form (i), therefore for applying HAM, we have

$$\begin{cases} r\underline{u}' + \underline{u}^2 = r^2, \\ \underline{u}(0) = 0, \\ (2-r)\overline{u}' + \overline{u}^2 = (2-r)^2, \\ \overline{u}(0) = 0 \end{cases} \tag{19}$$

Table 1. The results of example 5.1 at $r = \frac{1}{3}$

m	$\underline{v}_m$	$\lvert\underline{v}_m - \underline{v}_{m-1}\rvert$	$C_{\underline{v}_m, \underline{v}_{m-1}}$	$\overline{v}_m$	$\lvert\overline{v}_m - \overline{v}_m\rvert$	$C_{\overline{v}_m, \overline{v}_{m-1}}$
1	-9.7781119	9.444778532	6.992363341	13.77811277	12.11144609	6.767230662
5	-2.6670010	0.333667729	6.798962872	12.35589047	0.689223386	6.401066186
10	-2.4629982	0.000593089	4.743743069	12.31499167	0.000216843	4.184706702
⋮	⋮	⋮	⋮	⋮	⋮	⋮
14	-2.4630186	0.3630	1.531724583	12.31509394	0.117	0.915507417
15	-2.4630186	@o	-0.19917534	12.31509399	0.50	0.517567408
16	-	-	-	12.31509398	@0	-0.62856062

Table 2. The results of example 5.1 at $r=\frac{2}{3}$

m	$\underline{v}_m$	$\lvert\underline{v}_m-\underline{v}_{m-1}\rvert$	$C_{\underline{v}_m,\underline{v}_{m-1}}$	$\bar{v}_m$	$\lvert\bar{v}_m-\bar{v}_m\rvert$	$C_{\bar{v}_m-\bar{v}_m}$
1	-3.3890563	3.722389672	6.635062216	8.389056406	7.055723074	6.807089392
5	2.299832867	0.0335005507	6.199460093	9.811278707	0.477945039	6.425932888
10	2.462998334	0.0004376400	5.188647972	9.851993334	0.000032671	3.885601800
⋮	⋮	⋮	⋮	⋮	⋮	⋮
14	2.463018788	0.2806	2.296235368	9.852075150	0.34	0.818789070
15	2.463018791	0.37	0.412832057	9.852075188	0.38	0.863685719
16	2.463018797	0.54	0.575559355	9.852075187	0.1	0.070409377
17	2.463018796	@0	-0.02650064	9.852075188	@0	-0.08897000

Table 3. The results of example 5.2 at $r=\frac{1}{4}$

m	$\underline{v}_m$	$\lvert\underline{v}_m-\underline{v}_{m-1}\rvert$	$C_{\underline{v}_m,\underline{v}_{m-1}}$	$\bar{v}_m$	$\lvert\bar{v}_m-\bar{v}_m\rvert$	$C_{\bar{v}_m-\bar{v}_m}$
1	0.208333378	0.0416666711	6.066928151	1.458333364	0.291666698	6.670330983
5	0.190564027	0.0003530844	6.396439090	1.333948188	0.002471590	5.940507069
10	0.190399619	0.1567	3.804958122	1.332797329	0.000010966	3.466073050
15	0.190398563	0.2280	1.904830080	1.332789941	0.1590	2.050867269
⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	0.190398543	0.67	0.349162978	1.332789803	0.30	0.274529360
21	0.190398543	0.46	0.111802062	1.332789801	0.23	0.274529360
22	0.190398543	@0	-0.08897004	1.332789799	@0	-0.32753063

Table 4. The results of example 5.2 at $r=\frac{3}{4}$

m	$\underline{v}_m$	$\lvert\underline{v}_m-\underline{v}_{m-1}\rvert$	$C_{\underline{v}_m,\underline{v}_{m-1}}$	$\bar{v}_m$	$\lvert\bar{v}_m-\bar{v}_m\rvert$	$C_{\bar{v}_m-\bar{v}_m}$
1	0.625000008	0.1250000075	6.666047409	1.041666679	0.208333346	6.678915248
5	0.571692076	0.0010592532	6.873549568	0.952820126	0.001765422	6.618287907
10	0.571198850	4.2196531290	4.219653129	0.951998083	0.7834	4.319980391
⋮	⋮	⋮	⋮	⋮	⋮	⋮
19	0.571195627	0.33	1.128513940	0.951992710	0.60	0.644771707
20	0.571195625	0.17	1.070409377	0.951992708	@0	-0.22989917
23	0.571195623	0.2	0.070409377	-	-	-
24	0.571195623	@0	-0.02650063	-	-	-

The results of applying the HAM by the mentioned algorithm based on the CESTAC method on Equation(19) by selections $h_1,h_2=\frac{-1}{2}$ *and* $H_1\equiv\frac{1}{r},H_2\equiv\frac{1}{2-r}$ at t = 1 in the levels $r=\frac{1}{4},\frac{3}{4}$ are gathered in the Tables 3 and 4 respectively. We can observe the optimal number of terms and optimal values in the tables.

6. Conclusion

In this work, we proposed a reliable scheme based on the stochastic arithmetic and the CESTAC method to find the optimal solution of a given fuzzy ordinary differential equations obtained from the HAM at a given point. Forthis purpose, we proved a theorem to determine the accuracy of the HAM. Also, an algorithm was performed to illustrate the advantage of using the stochastic arithmetic in place of the floating-point arithmetic. In this case, in the termination criterion, we applied the concept of the informatical zero in place of a determine value like. Therefore, we are able to find the correct significant digits of the results, the optimal number of iterations, and validate the results. The applicability, efficiency and importance of the stochastic arithmetic was shown in the examples. Consequently, in order to rely the results, it is suggested to perform the program in new proposedarithmetic. As a development of this idea, the new algorithm of HAM based on the CESTAC method can be performed on any fuzzy partial differential equation.

7. References

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