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Pointwise (Approximate) Versions of Amenability and Connes Amenability with Application over some Algebras

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Abstract

Suppose A is a presumably if intransitive commutative Banach algebra in its second dual A^{**}. The paper studied several weaker definitions on amenability, Connes amenability including quotient pointwise amenability and approximate Connes amenability. Furthermore, the relations between them were discussed.

Keywords: Amenability, Connes Amenability, Pointwise Amenability, Quotient Banach Algebras, w*-Approximately Connes Amenability

1. Introduction and Preliminaries

The implication of amenability for Banach algebras was first introduced by B. E. Johnson in[7] and after the decades, the concept of Connes-amenability was presented by V. Runde in[14]. These two concepts have received substantial attention by mathematicians. A number of definitions and problems were presented for these two concepts, some of which were solved and some others are still under investigation and are unsolved. The concept of amenability dubbed and numerous reviews pointwise amenability and similar propertis Connes amenability by Ghahremani and Loy[6] and Mahmoodi[9] respectively. Kazemipour and Fozouni[8], by surveying the most important and recent papers on amenability, defined Quotient amenability and focused on its examples and its applications in Fourier Algebra. The first section in the present research was to examine amenability definitions along with applied propositions on approximate identity.

In the second section, pointwise amenability conditions were investigated, and by using certain lemmas and a number of proposition, it was combined with Quotient amenability to make pointwise Quotient amenability. Furthermore, based on[8], some examples and theorems were given. Suppose that A is a second dual of Banach algebra. The third section was to review the notion of Connes amenability and to present concept of approximate Connes-amenability. Also, regarding Arens regular Banach algebras, theorem and proposition were introduced. Then, according to previous sections, the relation between Banach algebras A and $M_n(A)$ and their ideals was discussed.

The following theorem is one of the important characterization of amenability due to B. E. Johnson.

Theorem 1.1: [7]Let A be a Banach algebra, then it is amenable if and only if it has a bounded approximate diagonal.

Let A be a C∗-algebra. Then A is nuclear if for all C∗-algebra B, there exists only one C_∗-norm on $A \otimes B$ [16, Definition 6.3.4]. To see a proof of the following theorem see [16, Corollary 6.5.12].

Theorem 1.2: (Connes-Haagerup) Suppose that A be a C∗-algebra. Then A is amenable if and only if $A^{**}$is Connes amenable if and only A is nuclear.

Proof: See [16, Corollary 6.5.12 (i), (ii), (iii)].

Proposition 1.3: Suppose that A be a Arens regular Banach algebra. Then closed subalgebra of A quotient of

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A by closed ideals, and Arens-Hoffman extension of A are also Arens regular.

Proof: See [1, Corollary 2.6.18].

Definition 1.4: [1]A Banach algebra A is mentioned to be dual if there is a closed submodule A_* of A^* such that $A = (A_*)_*$.

If A is a dual Banach algebra, the predual module A_* need not be unique. In this paper, however, it is always clear, for a dual Banach algebra A, to which A_* we are referring. In particular, we may discussion of the w^*-topology on A sans ambiguity. We note a few elementary properties of dual Banach algebras:

Proposition 1.5: [13]Suppose that A be a dual Banach algebra. Then:

(a) Multiplication in A is separately w^*-continuous.
(b) A has an identity if and only if it has a bounded approximate identity.
(c) The Dixmier projection :

$$A^{**} \cong A_*^{**} \to A_*^* \cong A$$

is an algebra homomorphism with respect to either Arens multiplication on A^{**}.

Proof: (a) and (b) are obvious, and (c) follows from (a) and [12, Theorem 1].

Defnition 1.6: [16,15]Let A be a Banach algebra. A Banach A-bimodule E is called pseudo unital if $E = \{a.\, x.\, a\colon a, b \in A, x \in E\}$ Similarly, one defines psedo-unital left and right Banach modules.

Definition 1.7: [12]A [bounded] left approximate identity for A is a [bounded] net (u_α) such that $||a - u_\alpha a|| \to 0$ for each $a \in A$; similarly for right and two-sided approximate identities. By Cohens factorization theorem [1, Theorem 2.9.24], A factors whenever it has a bounded left or right approximate identity.

Remark 1.8: From a pointwise perspective, A has bounded approximate units if there is a constant $K > 0$ such that, for each $a \in A$ and $\in\, > 0$, there is $u \in A$ such that $//u// \leq K$ and $||a - au||+||a - ua|| < \in$

It is standard that this implies that A has a bounded approximate identity [1, Section 2.9].

Proposition 1.9: Suppose that A be a Banach algebra with a bounded right approximate identity, and $a_0 \in A$ and let E be a Banach A-bimodule such that $A.\, E = \{0\}$. Then the derivation D:A→E^* is pointwise inner at a_0, that is, there exists $x \in E^*$ such that $D(a_0) = a_0.x - x.\, a_0$

Proof: It is obvious that $E^*.\ A = \{0\}$, suppose that D: $A \to E^*$ be a derivation. Then it indicate that $D(ab) = a.D\,(b)\ (a,b \in A)$. Let (e_α) In accordance with the de_nition 1.7 be a bounded right approximate identity for A, and suppose that $\varnothing \in E^*$ be a w^*-accumulation point of (De_α). Without loss of totality, we may suppose that $\varnothing = w^*\!-\!lim_\alpha De_\alpha$. It indicate that

$$D_\alpha = \lim_\alpha D\,(ae_\alpha) = \lim_\alpha a.\, D(e_\alpha) = a.\varnothing(a \in A)$$

So that $D = ad_\varnothing$. Now let $x \in E^*$ then $D(a_0) = ad_x(a_0)$ it follows $D(a_0) = a_0\, .\ x - x.\ a_0$ that means D is pointwise inner.

Proposition 1.10: For a Banach algebra A with a bounded approximate identity the following are equivalent:
(a) D: $A \to E^*$ is pointwise inner for each Banach algebra A-bimodule E.
(b) D: $A \to E^*$ is pointwise inner for each psuedo-unital Banach algebra A-bimodule E.

Proof: Obviously, (a)⇒(b) holds. For the converse, let D: $A \to E^*$ be a continuous derivation. let $a_0 \in A$ and let $E_0 = \{a.\, x.\, a\colon a, b \in A, x \in E\}$ By Cohens factorization theorem. E_0 is closed submodule of E. Let $\pi : E_0 \to E_0^*$ be the restriction map .

It is routinely checked that π is a module homomorphism, so that $\pi 0D$: $A \to E_0^*$ is a derivation. Then at $a_0 \in A$ there is $\varnothing \in E^*$ such that $\pi 0D(a_0) = ad_{\varnothing o}(a_0)$. choose $\varnothing_o \in E^*$ such that $\varnothing | E_0 = \varnothing_o$. It follows that $\tilde{D} := D - ad_\varnothing : A \to E\dfrac{1}{0}$.

We have $E\dfrac{1}{0} \cong (E/E_0)*$ (as Banach$A$- bimodule). From the definition of E_0, it follows that A.(E/E_0) =\{0\}, so that $D : A \to E\dfrac{1}{0}$ is pointwise inner at a_0 by proposition 1.9 . Hence, there is $\psi \in E_0$ such that $\tilde{D}(a_0) = ad_\psi$ i.e. $D(a_0) = ad_\varnothing - \psi(a_0)$

Let G be a locally compact group. We say that G is an amenable group if there exists a positive functional $m \in L^1(G)^{**}$ with $||m|| = 1$ and $m(L_x f) = m(f)$ for all $x \in G$ and $f \in L^1(G)^*$

The class of amenable groups includes all compact groups and abelian groups, however the discrete free group F2 on two generators is not amenable. The following theorem is due to B. E. Johnson and its proof is in [7, Theorem 2.5]. This theorem is fundamental in the theory of amenable Banach algebras.

Theorem 1.11: (Johnson) Let G be a locally compact group. Then $L^1(G)$ is an amenable Banach algebra if and only if is an amenable group.

2. Amenability and Pointwise Amenability Preliminaries

Below we offer a few basic definitions:

Definition 2.1: [16]Suppose that A be a Banach algebra, then A is amenable if for each Banach A-bimodule E, every continuous derivation D:A→E^* is inner, that is, there exist $x \in E^*$ such that $D(a) = a.\ x.\ x.\ a(a \in A)$.

Definition 2.2: [5]A is approximately amenable if, for each Banach A-bimodule, every continuous derivation D:A→E^* is approximately inner, that is ,there exists a net (x_n)in E^* such that

$$D(a) = \lim_{n\to\infty}(a.\,x_n - x_n.a)(a \in A)$$

Definition 2.3: [3]Suppose that A be a Banach algebra. then:

(i) A is pointwise amenable at $a_0 \in A$ if for each Banach A-bimodule E, every continuous derivation D:A→E^* is pointwise inner at a_0, that is, there exist $x \in E^*$ such that

$$D(a_0) = a_0 : x - x\text{:}\ a_0.$$

(ii) A is pointwise approximately amenable at $a_0 \in A$ if for each Banach A-bimodule E, every continuous derivation D:A→E^* is pointwise approximately inner at a_0, that is, there exist a sequence $(x_n) \subseteq E^*$ such that

$$D(a_0) = \lim_{n\to\infty}(a_0.\,x_n - x_n a_0)$$

(iii) A is pointwise [approximately] amenable if A is pointwise [approximately] amenable at a_0 for each $a_0 \in A$.

Proposition 2.4: Suppose that A be an pointwise amenable commutative Banach algebra. Then A has a bounded approximate identity.

Proof: See ([9],[4]).

Corollary 2.5: By 2.1 it is clear that every amenable Banach algebra is pointwise amenable Banach algebra. Hence for example, if G is a locally compact group, then $M(G)$ is pointwise amenable if and only if G is a discrete and amenable group.

We will continue this section with an important theorem of the amenability of Banach algebra that motivated us for the main definition of this paper.

Theorem 2.6: Suppose that A be an amenable Banach algebra and I be a closed two sided ideal of that.
Then we have

1. The quotient Banach algebra $\dfrac{A}{1}$ is amenable.
2. If I has a bounded approximate identity, then it is amenable.

Proof: See [16, Corollary 2.3.2, Theorem 2.3.7].

Now we define our core as follows. This definition will be used to acquire amenable Banach algebras from non-amenbale Banach algebras.

Definition 2.7: Suppose that A be a Banach algebra and I be a non-zero closed two sided ideal of A. We say A is I-quotient pointwise amenable if, the quotient Banach algebra A/I is pointwise amenable and say A is quotient pointwise amenable if, for all closed two sided non-trivial ideal I of A, the Banach algebra A is I-quotient pointwise amenable.

By 2.3 and 2.1 every amenable Banach algebra is an I-quotient pointwise amenable Banach algebra for all closed two-sided ideal I, but there exists I-quotient pointwise amenable Banach algebras which are not amenable. Recall that for Banach algebra A, $\Delta(A)$ denote the space of all characters of A,that is, all non-zero linear and multiplicative map from A into c. The following examples is like [pointwise Versions] the examples[8, Examples 2.3 and 2.4].

Example 2.8: Suppose that A and B be Banach algebras, which A is not amenable and B is pointwise amenable. Also, let $\theta \in \Delta(B)$. Then the θ-Lau product, denoted by =A $\times_\theta$ B, that defined as the set $A \times B$ equipped with the multiplication

$$(a,b)(a',b') = (aa' + \theta(b)a' + \theta(b')a, bb')$$
$$(a,a' \in A, b,b' \in B) \tag{2.1}$$

and the norm, $\|(a,b)\| = \|a\| + \|b\|$ is a Banach algebra. We know A is a closed two-sided ideal of C and $C/A \cong$

B as Banach algebras. So, C/A is an pointwise amenable Banach algebra, but C is not pointwise amenable, because $C = A \times_\theta B$ is a strongly splitting extension of B, it is pointwise amenable if and only if both A and B are pointwise amenable, see[10].

Example 2.9: Suppose that G be a locally compact group which is not amenable and I be a closed two sided ideal of $L^1(G)$ with finite codimension. By [1, Corollary 3.3.27], $L^1(G)/I$ is ∗ - isomorphic to a finite dimensional C∗-algebra. Therefore, by considering presentation pointwise of [11, Theorem 6.3.9] and Connes-Haageroup's Theorem, $L^1(G)$ is I-quotient pointwise amenable, but by Johnson's Theorem $L^1(G)$ is not pointwise amenable.

Example 2.10: Because if A be a finite dimensional Banach algebra and be pointwise amenable then it has an identity, so if we present an finite dimensional Banach algebra that has not identity then it is not pointwise amenable. Now note that all finite dimensional Banach algebras are not pointwise amenable, as respects finite dimensional pointwise amenable Banach algebras are unital. Also each unital Banach algebra necessarily is not pointwise amenable. To see an example consider $L^1(F_2)$ for which F_2 is the discrete free group on two generators that is not pointwise amenable. So, $L^1(F_2)$ is unital but it is not pointwise amenable.

The pursuant theorem describe I-quotient pointwise amenability when I has a bounded approximate identity.

Theorem 2.11: Let A be a Banach algebra and I be a closed two sided ideal with a bounded approximate identity. Then the following are equivalent;

1. The Banach algebra A is I-quotient pointwise amenable.
2. For each Banach A/I*-bimodule* X, every continuous derivation D:A→E^* is pointwise inner.

Proof: (1⇒2): Let A be a I-quotient pointwise amenable at a_0 and let X be a Banach A/I-bimodule and D:A→E^* be a continuous derivation. Define the map $\tilde{D}: A/I \to X^*$ as follows

$$\tilde{D}(a_0 + I) = D(a_0) \tag{2.2}$$

Now show that $\tilde{D}$ is well-defined. Let $a, b \in A$ and $a + I = b + I$, so $a–b \in I$. By Cohen Factorization Theorem we have $I = I^2$. Therefore, there exists $c, d \in I$ such that $a–b = cd$. Since D is a derivation we have

$$D(a\text{-}b)=D(cd)=D(c).d+c.D(d) \tag{2.3}$$

But, for every $x \in X$ we have

$$<D\,(c).d,\, x>=<D\,(c),d\ o\ x> = <D\,(c),(d+I).x> =0 \tag{2.4}$$

Correspondingly, $< c.\ D(d),\ x>=0$. Hence, $(a–b)=0$. So, $\tilde{D}(a+I) = \tilde{D}(b+I)$ Now, if we show that $\tilde{D}$ is a inner pointwise derivation, I-quotient pointwise amenability of A yield that D is inner pointwise and this complete the proof. Let $a, b \in A$ and $x \in X$. Then we have,

$$\begin{aligned} &< \tilde{D}(a+I).(b+I), x > \\ &+ < (a+I).\tilde{D}(b+I), x > \\ &=< d(a).(b+I), x > \\ &+ < (a+I).D(b), x \geq < D(a), box > \\ &+ < D(b), x\, 0\, a > =< D(a).\, b, x > \\ &+ < a.D(b), x > =< d(ab), x > \end{aligned}$$

Therefore, eD is a inner pointwise derivation from $A{=}I$ into X^*.

2⇒1): Let D: A/I→ X^* be a continuous derivation where X is a Banach A→I-bimodule.

Define $\tilde{D}$: A→X^* by, $\tilde{D}\,(a) = D(a+1)$for all $a \in A$. In view of the definition of module actions of X, the map $\tilde{D}$ is a continuous derivation. Now from the hypothesis we conclude that D is inner.

Hence, A is I-quotient amenable.

The recent theorem give us a criterions of I-quotient pointwise amenability when I has a bounded approximate identity. Now we give a theorem which give a sufficient condition for I-quotient pointwise amenability without any conditions on I.

Theorem 2.12: Suppose that $((I\frac{1}{A})^*, □)$ be an pointwise amenable Banach algebra, for closed non-trivial two sided ideal I of A. Then A is I-quotient pointwise amenable when

$$I\frac{1}{A} = \left\{ f \in A^* : f(a) = 0\,(a \in I) \right\}$$

Proof: See [8, Theorem 2.7].

Now we compare the quotient pointwise amenability of the Banach algebra A by two non-zero closed two sided ideals $I \subseteq J$.

Theorem 2.13: Let I, J be two closed two sided ideals of A where $I \subseteq J$. Then

1. A is J-quotient pointwise amenable if A is I-quotient pointwise amenable.

2. A is I-quotient pointwise amenable if A is J-quotient pointwise amenable and J is I-quotient pointwise amenable.
3. J is I-quotient pointwise amenable if it has a bounded approximate identity and A is I-quotient pointwise amenable.

Proof: See [8, Theorem 2.11].

The approximate Quotient amenability simply occurs when if "$\frac{A}{1}$ is pointwise amenable" standard was substituted by "$\frac{A}{1}$ is approximate amenable" in defining 2.7. It is expected that its theorem and example are like section 2 theorem and examples. therefore, they are not considered here.

3. Connes Amenability and *w**-approximately Connes Amenable Preliminaries and Relations

Definition 3.1: [13]A dual Banach algebra A is Connes amenable if every $w\_$-continuous derivation from A into a normal, dual Banach A–bimodule is inner.

Let $A = (A_*)^*$ be a dual Banach algebra and let E be a Banach A–bimodule. We write $\sigma wc(E)$for the set of all elements $x \in E$ such that the map

$$A \to E, a \to \begin{cases} a\,.\,x \\ x\,.\,a \end{cases},$$

are w^*-weak continuous. The space $\sigma wc(E)$ is a closed submodule of E. It was shown in [16,Corollary 4.6], that $\pi^*(A_*) \subseteq \sigma wc\,(A\hat{\otimes}A)^*$.

Taking adjoint, we can extend π to an A-bimodule homomorphism $\pi_{\sigma wc}$ from to $\sigma wc\,((A\hat{\otimes}A)^*)^*$ to A.

Definition 3.2: A dual Banach algebra A is W^*-approximately Connes amenable if, for everynormal, dual Banach A–bimodul E, every W^*–continuous derivation $D{:}A\to E$ is W^*-approximately inner, that is, there exist a sequence$(xn) \subseteq E$ such that $D(a) = W^*\!-\!limn(a.\text{ø}n-\text{ø}n\_a)$.

Proposition 3.3: Permit A be a pointwise amenable, commutative Banach algebra, and let I be a Weakly complemented, closed ideal in A. Then I has a bounded approximate identity.

Proof: See [[3], proposition 1.6.3].

Proposition 3.4: Permit A be a pointwise amenable commutative Banach algebra.Then A has a approximate identity.

Proof: Immediate from the [[3], Proposition 1.6.3] and put $I = A$.

Proposition 3.5: Permit A be an Arens regular commutative Banach algebra which is an ideal in A^{**}. Then A^{**} is Arens regular commutative Banach algebra.

Proof: Permit A be a commutative Banach algebra. by $\varphi = \varphi\Diamond\text{ø}(\text{ø};\ \varphi \to A)$; $\text{ø} \in A$: $\text{ø}\ \square\ \varphi = \psi\ \varphi(\psi \in A^{**})$, we have A is Arens regular if and only if $(A^{**};\square)$ is commutative, hence $(A^{**},\Diamond)$ is also commutative.

Let A be a Banach algebra. From [4], we recall that A has left (right) approximate units if, for each $a \in A$ and $\varepsilon > 0$, there exists $u \in A$ such that $||a{-}ua|| < \in (||\ a{-}au|| <\in)$, and A has approximate units if, for each $a \in A$ and $\varepsilon > 0$, there exists $u \in A$ such that $(||a{-}ua||{+}||\ a{-}au||{<}\in)$.

The approximate units have bound m if the element u can be chosen such that $||u||{\le}m$. The Algebra A has a *bounded* (left or right) approximate units if it has (left or right) approximate units of bound m for some $m \ge 1$.

Definition 3.6: A dual Banach algebra $A{=}(A_*)^*$ has *left (right) w^*-approximate units* if, for each $a \in A$ and $\varepsilon > 0$, and for each finite subset $K \subseteq A_*$, there is $u \in A$ such that $\left|\left\langle \psi, a - ua\right\rangle\right| < \varepsilon \ \left(\left|\left\langle \psi, au - a\right\rangle\right| < \varepsilon\right)$ for $\in K$. We say A has w^* -approximate units if, for each $a \in A$ and $\varepsilon > 0$ and for each finite subset $K \subseteq A_*$, there is $u \in A$ such that $\left|<\psi, a - au>\right| + \left|<\psi, -ua>\right| <\in (\psi \in k)$. The appropriate w^*-approximate units have bounded m if the element u can be chosen such that $||u||{\le}m$. The dual Banach algebra A has *bounded* (left or right) w^*-approximate units if it has (left or right) w^*-approximate units of bound m for some $m \ge 1$.

The following elementary lemma is useful in considerations of identities.

Lemma 3.7: Suppose that A be a Banach algebra and take $m \ge 1$. Suppose that A has pointwise left identity of bound m(i.e: for every $a \in A$ there exists $u \in A$ *with* $||u|| \le m$ such that $ua = a$). Then, for each $a_1; \ldots;$ an $\in A$ there exists $u \in A$ with$||u||{\le}\ m$ such that $uai = ai$; $1{\le}\ i \le n$.

Proof: Take $a1;\ldots;an \in A$. Successively choose $u1;\ldots;Un \in A$ with $||ui|| \le m, 1 \le i \le n$ and $(e{-}ui)\ldots(e{-}u1)ai = 0\ (1 \le i \le n)$.

Define $u \in A$ by $e-u = (e-Un)$. Then for each $1 \leq i \leq n$ we have $ai-uai = (e-Un)\ldots (e-Ui+1)((e-ui)\ldots(e-u1)ai) = 0$ as required.

Theorem 3.8: Let A be an Arens regular commutative Banach algebra which is an ideal in A^{**}.

Then the following are equivalent :

(i) A is pointwise amenable.
(ii) A^{**} is w^*-approximately Connes amenable .

Proof: (i)⇒(ii) by using Theorem 1.2 and Proposition 3.4 is valid. (ii)⇒(i) since A^{**} is w^*-approximately Connes amenable, it has an identity by Proposition 3.4 . By [[4], Proposition 5.18] and Lemma 3.7, this means that A has bounded approximate identity, say (e_α). By proposition 1.10, it is therefore sufficient for A to be pointwise amenable that D: $A \to E^*$ is pointwise inner for each pseudo unital Banach A–bimodule.

Let E be a pseudo unital Banach A–bimodule, and let D: $A \to E^*$ be a derivation. By [[16], proposition 2.1.6], the bimodule action of A on E^* extends canonically to A^{**} and D has a unique extension $\tilde{D}$ has a unique extension $\tilde{D} \in Z^1(A^{**}, E^*)$. We claim that E^* is a normal, dual Banach algebra A^{**}-bimodule.

Let (a_α) be a net in A^{**} such

That $a_\alpha \to w^*$ 0, let, and let $x \in E$. Since E is pseudo-unital, there are $b \in A$ and $y \in E$ such that $x = y.b$. Since the W^*- topology of A^{**} restricted to A is the weak topology, we have $ba_\alpha \to w$ 0. So that $x.a_\alpha = y.ba_\alpha \to w$ 0 and consequently $<x, a_\alpha.ø> = <x. a_\alpha, ø> \to o$. Since $x \in E$ was arbitrary, this means that $a_\alpha.ø \to w^*0$. Analogously, one shows that ø. $a_\alpha \to w^*0$.

To see

That $\tilde{D}$ is w^* -continuous, again let$(a_\alpha)_\alpha$ be a net in A^{**} such that $a_\alpha \to w^*0$, let $x \in E$, and let $b \in A$ and $y \in E$ such that $x=b.y$ Then we have : $<x, \tilde{D}\ a_\alpha> = < b.y, \tilde{D}\ a_\alpha> = <y, (\tilde{D}\ a_\alpha).b> = <y, D(a_\alpha b) - a_\alpha.Db> \to 0$. Because D is weakly continuous and E^{**} is normal, dual Banach A^{**}–bimodule. From the w^*-approximately Connes amenability of A^{**} we conclude that $\tilde{D}$, and hence D, is pointwise inner.

We shall also need the following observation.

Corollary 3.9: Let A be a dual Banach algebra and I be a non-zero closed two sided ideal of .
Then:

(*i*) $M_n(A)$ is $M_n(I)$-quotient amenable if and only if A is I-quotient amenable;
(*ii*) $M_n(A)$ is $M_n(I)$-quotient pointwise amenable if and only if A is I-quotient pointwise amenable;
(*iii*) $M_n(A)$ *is* w^*-approximate Connes amenable if and only if A is w^*-approximate Connes amenable.

Proof: Let I be a non-zero closed two sided ideal of A then it is clear that $M_n(I)$ is a non-zero

Closed two sided ideal of $M_n(A)$. We modify slightly the proof of[[3], Proposition 1.6.7], as follows:

(*i*) This is corollary of [[8], Theorem 2.6].
(*ii*) This is Theorem 2.11 with [[2], Theorem 2.7(i)].
(*iii*) Suppose that $M_n(A)$ is w^*- approximately Connes amenable. Then we follow the proof of [[2] , Theorem 2.7(i)], replacing $\wedge$ by a suitable net $(\wedge_\upsilon)$, to see that A is w^*-approximately Connes amenable. Conversely, suppose that A is w^*-approximately Connes amenable. We modify slightly the proof of [[2], Theorem 2.7(ii)] by replacing the net (u_α) by the net given to us by [[6], Corollary 2.2] (and ignoring the bound estimates).

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