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Weighted Peripheral Graph

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Abstract

Weighted peripheral graph G_d of a graph G has the peripheral vertices of G as its vertices and the diametral paths between the peripheral vertices of G as its edges. The structural properties of this weighted graph for some classes of graphs are discussed and bounds of certain parameters are identified. For the C# program developed to determine the parameters involved in the study, corresponding output for a sample graph is also presented. Remote nodes, high priority routes between them and strategic location problems of real life networks are some areas where these results can be applied.

Keywords: Central Vertex, Diameter, Diametral Path, Peripheral Vertex, Radius, Weighted Graph, Weighted Peripheral Graph

Mathematics Subject Classification (2010): 05C12

1. Introduction

The peripheral vertices and diametral paths play an important role in analysing and designing of networks. The study on the peripheral vertices helps us solve problems in transportation, distribution, designing, communication, team formation and event management. Researchers have worked on paths, diameter and diametral paths in literature[1–3]. The concept of a graph in which every induced subgraph has a pair of dominating peripheral vertices called diametral path graph has been introduced in[4].

In this paper, a study on peripheral vertices and diametral paths is undertaken and a weighted graph is generated. In Section 2, a weighted Graph named weighted peripheral graph G_d constructed from a simple, connected, undirected and unweighted graph G is introduced. Also results on the structural properties of G_d for a few classes of graphs are discussed. In Section 3, a sample graph is given with the corresponding output for the software program developed in C# to determine the parameters involved in this study.

The definitions and results are in accordance with[5–7]. The length of a path is the number of edges on the path. The distance between two vertices in a graph is the length of shortest path between them. The eccentricity of a vertex is the maximum of distances from it to all the other vertices of that graph. While diameter is the maximum of the eccentricities of all vertices of that graph, the radius is minimum of these. Peripheral vertices are vertices of maximum eccentricity and central vertices are of minimum eccentricity. The diametral path of a graph is the shortest path between two vertices which has length equal to diameter of that graph.

Given below are a few standard results in certain classes of graphs:

1. Complete Graph K_n: $Diam(K_n) = 1$ where $n \geq 2$.
2. Wheel W_n: $Diam(W_n) = 2$ where $n \geq 5$.
3. Star $K_{1,n}$: $Diam(K_{1,n}) = 2$ where $n \geq 2$.
4. Complete Bipartite Graph $K_{m,n}$: $Diam(K_{m,n}) = 2$ where $m \geq 2$ or $n \geq 2$.
5. Path P_n: $Diam(P_n) = n-1$ and $Rad(P_n) = \lfloor n/2 \rfloor$.
6. Cycle C_n: $Diam(C_n) = \lfloor n/2 \rfloor$ where $n \geq 3$.

2. Bounds

2.1 Definition

Weighted peripheral graph G_d is a weighted graph that is generated from the given graph G by representing peripheral vertices of G as vertices of G_d and diametral paths

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between peripheral vertices in G as edges between vertices in G_d. The weight of an edge in G_d is represented by the number of diametral paths between the corresponding pair of vertices in G.

2.2 Example

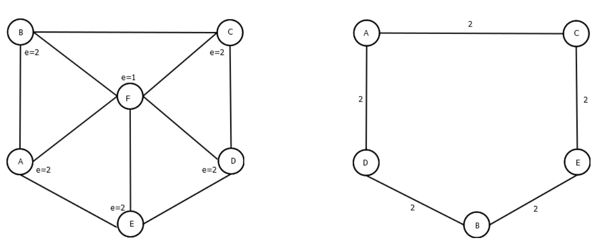
Consider a graph G given in the Figure 1(a). We can note that Diam(G) = 2. The peripheral vertices are A, B, C, D and E. The diametral paths are ABC, AFC, BCD, BFD, CDE, CFE, DEA, DFA, EAB, EFB. Hence weighted peripheral graph G_d has vertices A, B, C, D and E and edges AC, BD, CE, AD, BE with corresponding weights 2, 2, 2, 2 and 2 as observed in Figure 1(b).

2.3 Theorem

a. The weighted peripheral graph of a path P_n is an edge with weight 1 for all n.
b. The weighted peripheral graph of a star $K_{1,n}$ is a complete graph K_n with weight 1 for each edge.
c. The weighted peripheral graph of a complete graph K_n is a complete graph K_n with weight 1 for each edge.

Proof:

a. For a path P_n, there are only two peripheral vertices and one diametral path between them. Hence the weighted peripheral graph has two vertices with an edge between them with weight 1.
b. Consider star $K_{1,n}$ ($n \geq 2$). It has Diam($K_{1,n}$) = 2 and there are n peripheral vertices with a diametral path between every pair of those vertices. Hence the weighted peripheral graph of star $K_{1,n}$ has n vertices with an edge between every pair of vertices with weight 1and is a complete graph K_n.
c. Consider complete graph K_n ($n \geq 2$). Since Diam(K_n) = 1, every vertex is a peripheral vertex and every edge is a diametral path. Hence the weighted peripheral graph has all the n vertices and edge between every pair of vertices with weight 1. Hence the weighted peripheral graph is a complete graph K_n.



(a) (b)

Figure 1. **(a)** G. **(b)** G_d.

2.4 Theorem

a. The weighted peripheral graph of an odd cycle C_n is an odd cycle C_n with weight 1 for each edge.
b. The weighted peripheral graph of an even cycle C_n is a disconnected graph of n vertices with weight 2 for each edge.

Proof:

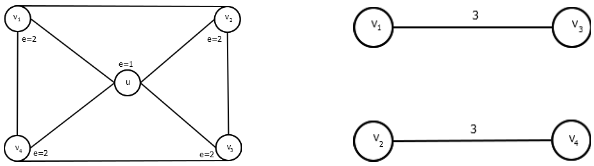
a. Consider an odd cycle C_n ($n \geq 3$). There are two diametral paths from each vertex with end vertices which are adjacent. In other words, any two adjacent vertices have diametral paths to a common end vertex. Hence the weighted peripheral graph has n vertices with two edges incident on each vertex and is an odd cycle C_n. Also weight of each edge is 1, as there is only one diametral path between a pair of vertices.
b. Consider an even cycle C_n ($n \geq 4$). Since n is even, Diam(C_n) = n / 2. Also from a vertex, there are exactly two diametral paths to a common end vertex. Any two diametral paths have same pair of end vertices or a different pair of end vertices and no two diametral paths have only one common end vertex. Hence the weighted peripheral graph has edges which do not share a vertex and is a disconnected graph. Also weight of each edge is 2, as there are only two diametral paths between a pair of vertices.

2.5 Theorem

a. The weighted peripheral graph of W_5 is a disconnected graph of 4 vertices with weight 3 for each edge.
b. The weighted peripheral graph of W_n ($n > 5$) is a connected graph of n-1 vertices.
c. The weighted peripheral graph of a complete bipartite graph $K_{r,s}$ is a disconnected graph with two connected complete components K_r and K_s. Also weight of each edge of K_r is s and of K_s is r.

Proof:

a. Consider Wheel W_5 in Figure 2(a). It has 4 peripheral vertices and Diam(W_5) = 2.
   Let v_1, v_2, v_3 and v_4 be the peripheral vertices and u be the central vertex. The diametral paths of W_5 are v_1uv_3, $v_1v_2v_3$, $v_1v_4v_3$, v_2uv_4, $v_2v_1v_4$ and $v_2v_3v_4$. Hence the weighted peripheral graph is disconnected with vertices v_1, v_2, v_3 and v_4. Also the edges are v_1v_3 and v_2v_4 with corresponding weights 3 and 3 as observed in Figure 2(b).
b. Consider Wheel W_n($n > 5$). It has n-1 peripheral vertices and Diam(W_n) = 2.



(a) (b)

Figure 2. **(a)** G. **(b)** G_d.

Let $v_1, v_2, v_3 \ldots v_{n-1}$ be the peripheral vertices of W_n. From v_1, there are diametral paths to v_3, v_4 and so on to v_{n-2} and no diametral paths to v_2 and v_{n-1}. Also from v_2, there are diametral paths to v_4, v_5 and so on to v_{n-1}. Hence the weighted peripheral graph has vertices $v_1, v_2, v_3 \ldots v_{n-1}$. Considering edges v_1v_3, $v_1 v_4 \ldots v_1v_{n-2}$, there is a path from v_1 to v_3, v_4, v_{n-2}. Also considering edges v_2v_4, $v_2v_5 \ldots v_2 v_{n-1}$, there is a path there is a path from v_2 to $v_4, v_5 \ldots v_{n-1}$. Considering edges v_1v_4, v_2v_4 and $v_2 v_{n-1}$ there is a path from v_1 to v_2 and v_{n-1}. Since there is a path from v_1 to all vertices, there is a path between every pair of vertices and the weighted peripheral graph is connected.

c. In a complete bipartite graph $K_{r,s}$ with partites V_1 and V_2, all diametral paths have both end vertices lying in the same partite and no diametral path has one end vertex in V_1 and the other in V_2. Since every pair of vertices of a partite have a diametral path between them, the weighted peripheral graph has two partites and edges between every pair of vertices in each partite. Hence the weighted peripheral graph is disconnected and has two complete components K_r and K_s. Since for every pair of vertices of a partite, diametral path passes through one of the vertices of the other partite. The number of diametral paths between a pair of vertices of a partite is the number of vertices of the other partite. Hence weight of each edge of K_r is s and of K_s is r.

2.6 Theorem

If m is the number of edges in G_d and p the number of peripheral vertices in G, then $\lceil p/2 \rceil \leq m \leq pC_2$.

Proof:
Since there is atleast one diametral path from each peripheral vertex in G, there is atleast one edge incident on every vertex in G_d. Since the minimum number of ways in which all vertices appear in some edge in G_d is $\lceil p/2 \rceil$, the least number of edges that are formed among p number of vertices in G_d is $\lceil p/2 \rceil$.

Hence $m \geq \lceil p/2 \rceil$.

Also if there is an edge between every pair of vertices, the maximum number of edges that can be formed among p number of vertices in G_d is pC_2.

Hence $m \leq pC_2$.

Hence $\lceil p/2 \rceil \leq m \leq pC_2$.

Remark:
We can clearly see that sum of weights of the edges in G_d is equal to the total number of diametral paths in G.

2.7 Proposition

If w_i is the weight of i^{th} edge in G_d and n(P) the total number of diametral paths in G, then $1 \leq w_i \leq n(P)$.

Proof:
An edge present between two peripheral vertices indicates there is atleast one diametral path between them in G.

Hence $w_i \geq 1$.

As n(P) is the total number of diametral paths in G, the weight of an edge cannot exceed n(P).

Hence $w_i \leq n(P)$

Hence we can conclude that $1 \leq w_i \leq n(P)$.

2.8 Definition

Peripheral degree of a vertex in G_d is the sum of the weights of the edges incident on it.

2.9 Proposition

If D_i is the peripheral degree of i^{th} vertex in G_d and n(P) the total number of diametral paths in G, then $1 \leq Di \leq n(P)$.

Proof:
As there is atleast one diametral path from each peripheral vertex in G, there will be atleast one edge incident on every vertex in G_d.

Hence $D_i \geq 1$.

As n(P) is the total number of diametral paths in G, the number of edges incident on a vertex in G_d cannot exceed n(P). Hence $D_i \leq n(P)$.

Hence $1 \leq D_i \leq n(P)$.

2.10 Proposition

If weighted peripheral graph G_d is a connected graph with diameter d and p number of vertices, then $1 \leq d \leq p$-1.

Proof:
As there are atleast two peripheral vertices in G, there will be at least one edge in G_d.

Hence $d \geq 1$.

If all vertices and edges of G_d form a path, then diameter is maximum and is p-1.

Hence $d \leq p$-1.

Hence we can conclude that $1 \leq d \leq p$-1.

2.11 Proposition

If weighted peripheral graph G_d is a connected graph with radius r and p number of vertices, then $1 \leq r \leq \lfloor p/2 \rfloor$.

Proof:
As there are atleast two peripheral vertices in G, there will be atleast one edge in G_d.

Hence $r \geq 1$.

If all vertices and edges of G_d form a path, then radius is maximum and is $\lfloor p/2 \rfloor$. Hence, $r \leq \lfloor p/2 \rfloor$.

Hence, we can conclude that $1 \leq r \leq \lfloor p/2 \rfloor$.

2.12 Proposition

Weighted peripheral graph G_d has no isolated vertex.

Proof:
Since there is a diametral path from every peripheral vertex in G, there is an edge from every vertex in G_d. Hence G_d cannot have an isolated vertex.

2.13 Proposition

If Diam(G) = 1, then weighted peripheral graph G_d is a connected graph.

Proof:
As Diam(G) = 1, all the vertices are peripheral and there is an edge between every pair of vertices in G.

Hence, every pair of vertices in G_d has an edge between them.

Hence G_d is a connected graph.

2.14 Definition

The edge in G_d that has highest weight is called strong link and the pair of vertices it is incident on are called strongly linked pair of vertices. The corresponding weight is called strong link number S.

2.15 Definition

The edge in G_d that has least weight is called weak link and the pair of vertices it is incident on are called weakly linked pair of vertices. The corresponding weight is called weak link number W.

Remark:
From **2.7 Proposition**, it can be noted that $1 \leq W \leq S \leq n(P)$.

3. Results

An algorithm is developed to determine

a. eccentricities, diameter, radius and diametral paths of any connected graph G.
b. vertices and edges with weights of weighted peripheral graph G_d.
c. diameter, radius and diametral paths of weighted peripheral graph G_d, if it is connected.

Consider a sample graph in Figure 3(a). Its corresponding output for the software program developed in C# to determine the parameters involved in this study is presented below.

Input: A-B; B-C; C-D; D-E; E-F; F-A; F-B; F-C
Output:

****************** InputGraph ******************

A-B B-C C-D D-E E-F F-A F-B F-C

****************** Results ********************

Eccentricities:
A: 3
B: 2
C: 2
D: 3
E: 2
F: 2

Radius: 2
Diameter: 3
Peripheral Vertices:
A
D
Central Vertices:
B
C
E
F

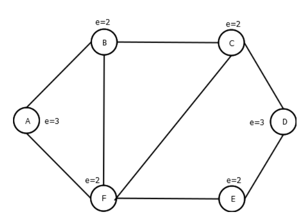


Figure 3. **(a)** Sample graph.

Total number of Diametral Paths: 3
Diametral Paths:

1. ABCD
2. AFED
3. AFCD

******** Weighted Peripheral Graph *********

Vertices:
A
D

Edge: Weight
A-D: 3

Eccentricities:
A: 1
D: 1

Radius: 1

Diameter: 1

Total number of Diametral Paths: 1

Diametral Paths:
1) AD

4. Conclusion

In this paper, a weighted graph named weighted peripheral graph G_d is introduced. In the future, the focus of study would be on the applications of these concepts.

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