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Interpolation of Fuzzy Data by Cubic and Piecewise-Polynomial Cubic Hermites

Hossein Vosoughi and Saeid Abbasbandy*

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran; abbasbandy@yahoo.com, hosein1204@gmail.com

Abstract

Background/Objectives: The purpose of this paper is to introduce fuzzy data cubic Hermite interpolation and develop same approach to present fuzzy-valued piecewise cubic Hermite interpolation. **Methods/Statistical Analysis:** It's done on the basis of linear space nota in sand by patching together a unique class of cardinal basis functions which satisfy a vanishing property on separated subintervals. **Findings:** We have presented a numerical method in full detail along with an explicit formula that comprises a simple way in order to calculate the results. Furthermore, some properties of new interpolants are provided, with a couple of computational examples to illustrate the mentioned method. **Application/ Improvement:** The presented procedure can be used rather than fuzzy simple Hermit and piecewise Hermite of order three interpolations, with exactly the same data, as an atternetive.

Keywords: Cardinal basis Functions, Fuzzy Data Interpolation, Piecewise Cubic Hermite Interpolation Ploynomia

1. Introduction

The problem of fuzzy interpolation was established by[1]. First solution to this problem has been proposed by[2] through the results of the Lagrange fundamental polynomial interpolation theorem. In[3] has formulated a calculational method for computing the fuzzy Lagrange interpolation and obtained a fuzzy data cubic spline interpolation according to some specific conditions. Moreover[4–8] have introduced fuzzy simple Hermite interpolation, E (3) fuzzy cubic splines, fuzzy splines, fuzzy complete splines and natural splines respectively. Then[9] have raised some fuzzy surfaces in light of the fuzzy Lagrange interpolating functions and fuzzy spline with the not-a-knot condition.

A simple method to interpolation of fuzzy data has presented by[10] that was consistent and inherited smoothness properties of the generator Hermite and piecewise cubic Hermite interpolation, but the method was expressed in a very special case and were not investigated for other three important remaining cases. This fact is a strong reason for the method weakness.

In total, a useful way to improve the interpolation and diminish the wiggling is using low order version of piecewise polynomial interpolants instead of higher orders, especially about widely used piecewise Hermite interpolation. In view of this fact with regard to the perspective of cardinal basis functions, we have interested to introduce piecewise-polynomial cubic fuzzy Hermite interpolation as an alternative to fuzzy simple Hermite interpolation in[4] and piecewise cubic fuzzy Hermite polynomial that was constructed by very weak conditions in[10].

In this paper, we have presented a method which its theoretic underpinning is linear space notions involves a set of piecewise cubic Hermite polynomials[11,12] where each of the mis unique and must vanish identically on some given intervals. For calculating the results of method, an explicit formula in a succinctly algorithm has been provided.

* *Author for correspondence*

2. Preliminaries

To start, we have reminded in brief some notions and basic concepts used throughout the paper. We will display the family of all nonempty convex, normal, upper semicontinuous and compactly supported subsets on $\mathbb{R}$ by $\mathbb{R}_f$. u is a fuzzy number iff $u \in \mathbb{R}_{\mathcal{F}}$.

$u^\alpha = \{x \mid x \in \mathbb{R} \,\&\, u(x) \geq \alpha\}, 0 < \alpha \leq 1,$

shows the α -cut of $u \in \mathbb{R}_{\mathcal{F}}$, which are closed bounded intervals $\mathbb{R}$ and denoted by $u^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$. $u^1 = \{x \mid x \in \mathbb{R} \,\&\, u(x) = 1\}$ is the core of u. For $a \leq b \leq c \leq d, u = < a, b, c, d >$ is a trapezoidal fuzzy number and a triangular fuzzy number is obtained when b = c. For α = [0,1] we have $u^\alpha = [a + \alpha(b-a), d - \alpha(d-c)]$.

Definition 2.1 When[8] R and L be continuous and decreasing functions from I = [0,1] to I which R(0) = L(0) = 1 and R(1) = L(1) = 0, an L - R fuzzy number u= (m, l, r)$_{LR}$ is a function from $\mathbb{R}$ into I satisfying

$$u(x) = \begin{cases} R(\dfrac{x-\alpha}{\beta}), & \alpha \leq x \leq a+\beta, \\ L(\dfrac{\alpha - x}{\alpha}), & a-\alpha \leq x \leq \alpha, \\ 0, & otherwise. \end{cases}$$

When[5] R (x) = L(x) = 1-x we have L - L fuzzy numbers that involve the triangular fuzzy numbers. The support of an L - L fuzzy number u= (m, l, r) will denote by the closed interval [m-l, m+r]. Based on the α - cuts, for all $0 \leq \alpha \leq 1, u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$ we have:

$$(u+v)^\alpha = u^\alpha + v^\alpha = \{x + y \mid x \in u^\alpha, y \in v^\alpha\}$$
$$(\lambda u)^\alpha = \lambda u^\alpha = \{\lambda x \mid x \in u^\alpha\}$$
$$(0)^\alpha = \{0\}$$

Throughout the paper, we will assume u is a triangular L - L fuzzy number.

Definition 2.2 Let $x_0, x_1, ..., x_n$ be n + 1 distinct points and $f_0, f_1, ..., f_n$ be associated function values. Specific real functions $\phi_j : \mathbb{R} \to \mathbb{R}, (j = 0,1,\ldots,n)$ called cardinal basis functions for which

$$\phi_j(x_i) = \sigma_{ij}, \quad i = 0,1,\ldots,n$$

(σ_{ij} is Kronecker's delta).

The set of ϕ_j 's is linearly independent and for each $1 \leq j \leq n-1, \phi_j(x)$, must vanish identically outside the interval $[x_{j-1}, x_{j+1}]$.

Definition 2.3 For the given cardinal basis functions $\phi_0, \phi_1, ..., \phi_n$, the function $F(x) = \sum_{j=0}^{n} f_j \phi_j(x)$ is defined as an interpolant and such an interpolation procedure is known as the cardinal basis functions method.

3. Fuzzy Cubic Hermite Interpolition Polynomial

Theorem 3.1 For[13] a given

$f \in C^1[x_1, x_2], x_1, x_2 \in \mathbb{R}, \quad (x_1 < x_2),$

there exists a unique cubic polynomial s_3 (x) solves the interpolation problem

$$s_3^{(k)}(x_i) = f^{(k)}(x_i), \quad k = 0,1, i = 1,2 \tag{1}$$

One of the ways to construct and denoting the existence of s_3 (x) is setting

$$s_3(x) = \sum_{i=1}^{2} f(x_i)\phi_i(x) + \sum_{i=1}^{2} f'(x_i)\psi_i(x)$$

and let

$$\phi_1(x) = \frac{(x-x_2)^2[(x_1-x_2)+2(x_1-x)]}{(x_1-x_2)^3} \tag{2}$$
$$\phi_2(x) = \frac{(x-x_1)^2[(x_2-x_1)+2(x_2-x)]}{(x_2-x_1)^3}$$
$$\psi_1(x) = \frac{(x-x_1)(x-x_2)^2}{(x_1-x_2)^2}$$
$$\psi_2(x) = \frac{(x-x_1)^2(x-x_2)}{(x_1-x_2)^2}$$

which satisfy the conditions

$$\phi_i(x_j) = \delta_{ij}$$
$$\phi_i'(x_j) = 0, \quad 1 \leq i, j \leq 2$$

and

$$\psi_i(x_j) = 0 \tag{3}$$
$$\psi_i'(x_j) = \delta_{ij}, \quad 1 \leq i, j \leq 2.$$

Theorem 3.2 Assume that $\phi_i(x), \psi_i(x), i = 1,2,$ are the well-known functions in Equation (2), then

- $\phi_1(x) + \phi_2(x) = 1, x \in [x_1, x_2],$
- For all $x \in [x_1, x_2]$, the sign of $\phi_i(x), i = 1,2,$ as well ψ_1 (x) is positive and ψ_2 (x) is negative.

Proof. The proof is trivial. á

Let I = [x_1, x_2], we would like to generate a cubic fuzzy-valued function as $s: I \to \mathbb{R}_{\mathcal{F}}$, such that $s^{(k)}(x_i) = f^{(k)}(x_i) = u_{ki} \in \mathbb{R}_{\mathcal{F}}$. Moreover, for $k=0,1,\ i=1,2,$ if $u_{ki} = y_i^{(k)}$ are crisp numbers in $\mathbb{R}$ and $f^{(k)}(x_i) = \chi_{y_i^{(k)}}$ then there is a cubic polynomial on I with $s^{(k)}(x_i) = y_i^{(k)}$ such that $s(x) = \chi_{f(x)}$ for all x $\in \mathbb{R}$, where $\{(x_i, f_i, f_i') \mid f_i^{(k)} \in \mathbb{R}_{\mathcal{F}}, k=0,1, i=1,2\}$ is given.

Through the extension principle results, membership function of s(x) for each x $\in$ [x_1, x_2] is as follows:

$$\mu_{s(x)}(t) = \sup_{\substack{y_1, y_2, y_1', y_2' \\ t = s_{y_1, y_2, y_1', y_2'}(x)}} \min(\mu_{u_{01}}(y_1), \mu_{u_{02}}(y_2), \mu_{u_{11}}(y_1'), \mu_{u_{12}}(y_2'));$$
$$= 0 \quad \text{if} \quad s^{-1}_{y_1, y_2, y_1', y_2'}(t) = \varnothing.$$

We assume such a cubic function exists and attempt to construct it in the light of[2]. According to[3,14], the α - cuts of s(x) follow a succinctly algorithm.

$$\begin{aligned} s^{\alpha}(x) &= \{t \in \mathbb{R} \mid \mu_{s(x)}^{(t)} \geq \alpha\} \\ &= \{t \in \mathbb{R} \mid \exists y_i^{(k)} : \mu_{u_{ki}}^{(y_i^{(k)})} \geq \alpha, k=0,1, i=1,2 \,\&\, s_3(x) = t\} \\ &= \{t \in \mathbb{R} \mid t = s_3(x) = s_{y_1, y_2, y_1', y_2'}(x), y_i^{(k)} \in u_{ki}^{\alpha}, k=0,1, i=1,2\} \\ &= \sum_{i=1}^{2} (u_{0i}^{\alpha}\phi_i(x) + u_{1i}^{\alpha}\psi_i(x)) \end{aligned}$$

Where,

$$s_{y_1, y_2, y_1', y_2'}(x) = \sum_{i=1}^{2} (y_i \phi_i(x) + y_i' \psi_i(x))$$

is a cubic Hermite polynomial in crisp case and by definition

$$s^{\alpha}(x) = \sum_{i=1}^{2} (u_{0i}^{\alpha}\phi_i(x) + u_{1i}^{\alpha}\psi_i(x)), \tag{4}$$

Therefore a formula that comprises a simple practical way for calculating s(x) is

$$s(x) = \sum_{i=1}^{2} (u_{0i}\phi_i(x) + u_{1i}\psi_i(x)). \tag{5}$$

Because of $u_{ki}^{\alpha} = [\underline{u}_{ki}^{\alpha}, \overline{u}_{ki}^{\alpha}]$, we will have $s^{\alpha}(x)$ by solving the following optimization problems:

$$\max \quad \& \quad \min \quad \sum_{i=1}^{2} (y_i \phi_i(x) + y_i' \psi_i(x)) \tag{6}$$
$$\text{subject to} \quad \underline{u}_{ki}^{\alpha} \leq y_i^{(k)} \leq \overline{u}_{ik}^{\alpha}, \quad k=0,1, i=1,2$$

from Theorem 3.2 these problems have the optimal solutions:

Maximization:

$$y_i^{(k)} = \begin{cases} \overline{u}_{ki}^{\alpha} & \text{if} \quad (k,i) \neq (1,2), \\ \underline{u}_{ki}^{\alpha} & \text{if} \quad (k,i) = (1,2), \end{cases} \qquad k=0,1, i=1,2$$

Minimization:

$$y_i^{(k)} = \begin{cases} \underline{u}_{ki}^{\alpha} & \text{if} \quad (k,i) = (1,2), \\ \overline{u}_{ki}^{\alpha} & \text{if} \quad (k,i) \neq (1,2), \end{cases} \qquad k=0,1, i=1,2$$

Theorem 3.3 If s(x) is an interpolating cubic fuzzy Hermite polynomial, then for all $\alpha \in [0,1]$

$$\text{len}\, s^{\alpha}(x) \geq \min\{\text{len}\, s^{\alpha}(x_1), \text{len}\, s^{\alpha}(x_2)\},$$

where we denote the length of an interval by len.

Proof. The proof is trivial.á

4. Fuzzy Piecewise Cubic Hermite Interpolation Polynomial

In this section we have focused on the piecewise case fuzzy polynomials and developed the previous approach based on the linear space notions, as the main purpose of the paper.

Let throughout $\Delta: a = x_0 < x_1 < \ldots < x_n = b,$ be a partition of I = [a,b] All piecewise cubic Hermite polynomials form a linear space will be designated by H_3 (Δ;I).

Definition 4.1 H_3 (Δ;I) is a set of all piecewise polynomial functions s(x) of degree at most three, defined on I such that $s(x) \in C^1[a,b]$. The function s(x) on $[x_{i-1}, x_i], 1 \leq i \leq n,$ defined by $s_i(x)$, an once continuously differentiable piecewise cubic Hermite polynomial on I.

Definition 4.2 Given[11] any real-valued function $f \in C^1(I)$.

Its H_3 (Δ;I) interpolate is the function s(x) of degree three on each interval $[x_{i-1}, x_i], 1 \leq i \leq n,$ such that

$$D^{(k)}s(x_i) = D^{(k)}f(x_i), \tag{7}$$
$$\text{for all} \quad k=0,1, 0 \leq i \leq n, D = \frac{d}{dx}$$

The proof of existence and uniqueness of full Hermite interpolation is given in[15], since on a partitioned interval, Presentation (7) is actually a special case of full Hermite interpolation, it follows, any function belongs to C^1 (I) owns a unique H_3 (Δ;I) interpolate[13].

$\mathcal{B}=\{\phi_{ik}(x)\}_{i=0,k=0}^{n,1}$, is a specific set of cardinal basis[16] for linear space H_3 (Δ;I) of dimension $2(n+1)$, and the basis function $\phi_{ik}(x)$ is defined by

$$D^l\phi_{ik}(x_j)=\delta_{kl}\delta_{ij},0\le k,l\le 1,0\le i,j\le n,D=\frac{d}{dx}. \tag{8}$$

All of the cardinal basis cubic functions $\phi_{ik}(x),k=0,1,0\le i\le n$, are computed in detail with explicit formulas and determined sign on every interval $[x_{i-1},x_i],0\le i\le n,$ in[13].

Theorem 4.3 Assume that $\phi_{ik}(x)$ satisfies the Equation (8), then

- $\phi_{i0}(x)+\phi_{i1}(x)=1,$ for all $x\in(x_i,x_{i+1}),i=0,1,\dots,n-1$.
- the sign of $\phi_{i1}(x)$ is negative on $(x_{i-1},x_i),i=1,2,\dots,n,$.
- the sign of $\phi_{ik}(x)$ is positive on $(x_i,x_{i+1}),k=0,1,i=0,1,\dots,n-1.$

Proof. With respect to explicit formulas of ϕ_{ik} 's the proof is so trivial.á

For[13,17] any $f(x)\in C^1(I)$, and its piecewise cubic Hermite interpolation s(x) ∈ H_3 (Δ;I), , an equivalent direct representation of s(x) in Equation (7), can be uniquely expressed

$$s(x)=\sum_{i=0}^{n}\sum_{k=0}^{1}f^{(k)}(x_i)\phi_{ik}(x). \tag{9}$$

As we stated earlier, we want to construct a piecewise case fuzzy-valued function as $s:I\to\mathbb{R}_{\mathcal{F}}$ such that $s^{(k)}(x_i)=f^{(k)}(x_i)=u_{ki}\in\mathbb{R}_{\mathcal{F}},k=0,1,1\le i\le n.$

Moreover[2], if for all $k=0,1,0\le i\le n,u_{ki}=y_i^{(k)}$ are crisp numbers in $\mathbb{R}$ and $f^{(x)}(x_i)=\chi_{y_i^{(k)}}$ then for a given set of interpolation data $\{(x_i,f_i,f_i')\,|\,f_i,f_i'\in\mathbb{R}_{\mathcal{F}},0\le i\le n\}$, there is a polynomial of degree three on $[x_{i-1},x_i],0\le i\le n$, with $s^{(k)}(x_i)=y_i^{(k)},k=0,1,0\le i\le n,$ such that $s(x)=\chi_{f(x)},x\in\mathbb{R}.$

With respect to[2], we assume such a fuzzy function exists and define s(x) according to the α - cuts as follows:

$$s^{\alpha}(x)=\{t\in\mathbb{R}\,|\,t=s_{\substack{y_0,\dots,y_n\\y_0',\dots,y_n'}}(x),\,y_i^{(k)}\in u_{ki}^{\alpha},k=0,1,0\le i\le n\},$$

where $s_{\substack{y_0,\dots,y_n\\y_0',\dots,y_n'}}(x)=s(x)$ in Definition (8) and more precisely (9).

By defining

$$s^{\alpha}(x)=\sum_{i=0}^{n}\sum_{k=0}^{1}u_{ki}^{\alpha}\phi_{ik}(x) \tag{10}$$

a computational fuzzy representation of \$s(x)\$ can obtain, that is

$$s(x)=\sum_{i=0}^{n}\sum_{k=0}^{1}u_{ki}\phi_{ik}(x). \tag{11}$$

Because of, $u_{ki}^{\alpha}=[\underline{u}_{ki}^{\alpha},\overline{u}_{ki}^{\alpha}],k=0,1,0\le i\le n,s^{\alpha}(x)$ solving the following optimization problems:

$$\begin{aligned}&\max\ \&\ \min && \sum_{i=0}^{n}\sum_{k=0}^{1}y_i^{(k)}\phi_{ik}(x)\\ &\text{subject to} && \underline{u}_{ki}^{\alpha}\le y_i^{(k)}\le\overline{u}_{ki}^{\alpha},\quad k=0,1,0\le i\le n,\end{aligned} \tag{12}$$

from Theorem 4.3 it is clear that the optimal solutions are:

Maximization:

$$y_i^{(k)}=\begin{cases}\overline{u}_{ki}^{\alpha} & \text{if}\quad \phi_{ik}(x)\ge 0\\ \\ \underline{u}_{ki}^{\alpha} & \text{if}\quad \phi_{ik}(x)<0\end{cases},k=0,1,0\le i\le n,$$

Minimization:

$$y_i^{(k)}=\begin{cases}\underline{u}_{ki}^{\alpha} & \text{if}\quad \phi_{ik}(x)\ge 0\\ \\ \overline{u}_{ki}^{\alpha} & \text{if}\quad \phi_{ik}(x)<0\end{cases},k=0,1,0\le i\le n,$$

Theorem 4.4 Suppose that s(x) is an interpolating function satisfies the piecewise cubic fuzzy Hermite polynomial condition then $\text{len }s^{\alpha}(x)\ge\min\{\text{len }s^{\alpha}(x_i),\text{len }s^{\alpha}(x_{i+1})\}$ for all $\alpha\in[0,1]$ and $x\in(x_i,x_{i+1}).$

Proof. From Theorem 4.3 Section (i), the proof Theorem 3.3 applies and the claimfollows.á

Theorem 4.5 For each x ∈ $\mathbb{R}$ the piecewise fuzzy-valued Hermite polynomial interpolation s(x) is a triangular L-L fuzzy number if also $u_{ki}=(m_{ki},l_{ki},u_{ki}),0\le k\le 1,0\le i\le n$ be such a fuzzy number.

Proof. The support of a triangular L-L fuzzy number $u=(m,l,r)$ is the closed interval [m - l, m+r] so for each x and u_{ki} we will have:

$$s(x)=\left(\sum_{i=0}^{n}\sum_{k=0}^{1}u_{ki}\phi_{ik}(x)\right) \tag{13}$$

$$=\Bigg[\sum\sum_{\phi_{ik}\geq 0}(m_{ki}-l_{ki})\phi_{ik}(x)$$

$$+\sum\sum_{\phi_{ik}<0}(m_{ki}+r_{ki})\phi_{ik}(x),$$

$$\sum\sum_{\phi_{ik}\geq 0}(m_{ki}+r_{ki})\phi_{ik}(x)+$$

$$\sum\sum_{\phi_{ik}<0}(m_{ki}-l_{ki})\phi_{ik}(x)\Bigg]$$

$$=\Bigg[\sum_{i=0}^{n}\sum_{k=0}^{1}m_{ki}\phi_{ik}(x)-(\sum\sum_{\phi_{ik}\geq 0}l_{ki}\phi_{ik}(x)$$

$$-\sum\sum_{\phi_{ik}<0}r_{ki}\phi_{ik}(x)),\sum_{i=0}^{n}\sum_{k=0}^{1}m_{ki}\phi_{ik}(x)$$

$$+(\sum\sum_{\phi_{ik}\geq 0}r_{ki}(x)\phi_{ik}(x)-\sum\sum_{\phi_{ik}<0}l_{ik}\phi_{ik}(x))\Bigg]$$

$$=(m(x)-l(x),m(x)+r(x))$$

Example 4.6 The support of a triangular L-L fuzzy number $u=(m,l,r)$ is the closed interval [m - l, m+r] Suppose that $(0,(-1,0.5,1),(-3,1,2)),(2,(-6.7,2,8.7),(0.5,2,2.5))$, are the interpolation data. In Figure 1 the solid lines denote the support, the bold one is the core of cubic fuzzy Hermite polynomial interpolation $s(x), x\in[0,2]$ and the dashed line is the 0.5-cut set of s(x).

The presented procedure can be used rather than fuzzy simple Hermite interpolation in[4] and the mentioned method in[10], with exactly the samedata, as an alternative.

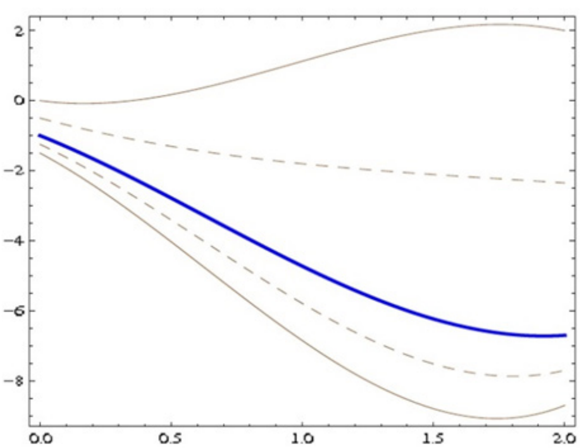

Figure 1. The results of Example (11).

Example 4.7 Consider[4] the numerical L-L fuzzy data

$(1,(0,2,1),(1,0,3)),(1.3,(5,1,2),(0,2,1)),$
$(2.2,(1,0,3),(4,4,3)),(3,(4,4,3),(5,1,2)),$
$(3.5,(0,3,2),(1,1,1)),(4,(1,1,1),(0,3,2))$

In Figure 2, the dashed line is the 0.5-cut set and the solid lines show the support and the core of fuzzy piecewise-polynomial cubic interpolation $s(x), x\in[1,4]$.

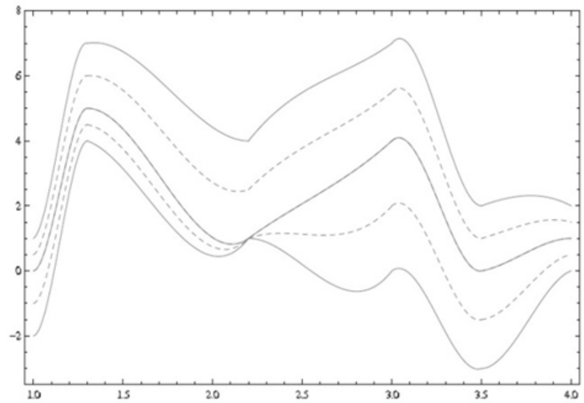

Figure 2. The results of Example 4.6.

5. Conclusion and Further Work

Based on a class of cardinal basis functions, which its theoretic substructure is linear space notions, we have successfully introduced a new low order fuzzy data interpolation and extended the results to practical fuzzy-valued piecewise-polynomial cubic Hermite interpolation in detail. The presented method in piecewise case could be applied as an alternative to fuzzy simple Hermite interpolation in[4] and the mentioned interpolation procedure of fuzzy data in[10] that was constructed with weak conditions without using the extension principle, contrary to our described method. The next step to improve the study is interpolation of fuzzy data, including switching points by a fuzzy piecewise polynomial interpolant in the general case and investigation an error estimation for the presented method.

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