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A New Approach based on Triangular Functions for Solving N-dimensional Stochastic Differential Equations

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Abstract

In this article, we prepare a new numerical method based on triangular functions for solving n-dimensional stochastic differential equations. At first stochastic operational matrices of triangular functions are derived then n-dimensional stochastic differential equations are solved recently. Convergence analysis and numerical examples are prepared to illustrate accuracy and efficiency of this approach.

Keywords: Brownian Motion, Itô Integral, N-dimensional Stochastic Differential Equations, Stochastic Operational Matrix, Triangular Functions.

AMS subject classifications. Primary: 65C30, 60H35, 65C20; Secondary: 60H20, 68U20

1. Introduction

Mathematical modeling of real world problems causes differential equations involving stochastic Gaussian white noise excitations. Such problems are modeled by stochastic differential equations(SDE). Some authors have presented numerical approachs to solve stochastic differential/integral equations[1–11]. We consider integral form of n-dimensional stochastic differential equation(N-SDE) as follows:

$$y(x) = y_0 + \int_0^x f(x,s)y(s)ds + \sum_{j=1}^{n}\int_0^x g_j(x,s)y(s)dB_j(s), \qquad x\in[0,1), \quad (1)$$

where, y_0 is an initial value, $B=(B_1(x),\ldots,B_n(x)$ is a n-dimensional Brownian process. $y(x)$ $f(x,s)$ and $g_j(x,s)$ $j=1,2,\ldots,n$; $s,x\in[0,T)$ are defined on (Ω,F,P), probability space, and $y(x)$ is unknown function. Also $\int_0^x g_j(x,s)y(s)dB_j(s), j=1,2,\ldots,n,$ are Itô integrals.

Orthogonal triangular functions (TFs) are derived from the Block Pulse Function (BPF) set by Deb et al.[12]. TFs approximation has been applied for the analysis of dynamical systems[13], integral equations[14,15] and integro-differential equations[16]

In Section 2, we review some properties of TFs. In Section 3, stochastic operational matrices of TFs are presented. Section 4 is devoted for solving N-SDE. In Section 5 is prepared convergence analysis of the approach. In Section 6 some numerical examples are provided. Finally, Section 7 gives a brief conclusion.

2. Brief Review of TFs

Deb[12] defined two m-set TFs over the interval [0,T) as follows

$$T1_i(x)=\begin{cases}1-\dfrac{1}{h}(x-ih) & ih\le x<(i+1)h,\\ 0 & o.w,\end{cases}$$

$$T2_i(x)=\begin{cases}\dfrac{1}{h}(x-ih) & ih\le x<(i+1)h,\\ 0 & o.w,\end{cases}$$

where, $i=0,\ldots,m-1$, and $h=\dfrac{T}{m}$.

TFs, are orthogonal,disjoint and complete[16].

M-set TF vectors can be considered as

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$T1(x)=[T1_0(x),...,T1_{m-1}(x)]^T$, $T2(x)=[T2_0(x),...,T2_{m-1}(x)]^T$,

and

$$T(x)=[T1(x),T2(x)]^T.$$

$\varphi(x)$, a square integrable function,may be approximated into TF series as:

$$\varphi(x) \simeq \hat{\varphi}(x) = \Phi1^T T1(x)+\Phi2^T T2(x)=\Phi^T T(x),\ x\in[0,T), \qquad (2)$$

where, $\Phi1_i=\varphi(ih)$ and $\Phi2_i=\varphi(i+1)h$ for $i=0,...,m-1$. The vectors $\Phi1$ and $\Phi2$ are called the 1D-TF coefficient vectors and 2m-vector Φ is defined as:

$$\Phi=[\Phi1,\Phi2]^T.$$

The operational matrix for integration can be obtained as [12]

$$\int_0^x T(s)ds = PT(x),$$

where,

$$P=\begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix},$$

and

$$P1=\frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{m\times m},\quad P2=\frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{m\times m}.$$

Let X be a $2m-$vector and B be a $2m\times 2m$ matrix; then, it can be concluded that

$$T(x)T^T(x)Y=\tilde{Y}T(x) \qquad (3)$$

and

$$T^T(x)BT(x)=\hat{B}T(x) \qquad (4)$$

in which $\tilde{Y}=diag(Y)$. Elements of $\hat{B}$,a $2m$ vector,are equal to the diagonal elements of B. Finally, integration of $\varphi(x)$ can be approximated as follows:

$$\int_0^x \phi(s)ds \simeq \int_0^x \Phi^T T(s)ds \simeq \Phi^T PT(x).$$

Any two variable function, $f(x,s)$, can be approximated by TFs as follows :

$$f(x,s)\simeq \hat{F}(x,s)=T^T(x)\,FT(s)$$

where,F is a $2m_1\times 2m_2$ coefficient matrix of TFs. We put $m_1=m_2=m.$ So, F can be expanded as:

$$F=\begin{pmatrix} F1 & F2 \\ F3 & F4 \end{pmatrix}_{2m\times 2m},$$

where, $F1$, $F2$, $F3$ and $F4$ are approximated by sampling $F(x,s)$ at points s_i and x_i such that $s_i=x_i=ih,$ for $i=0,1,...,m.$ So, the following approximations can be obtained

$$(F1)_{ij}=f(s_i,x_j)\quad i,j=0,1,...,m-1,$$

$$(F2)_{ij}=f(s_i,x_j)\quad i=0,1,...,m-1,\ j=1,...,m,$$

$$(F3)_{ij}=f(s_i,x_j)\quad i=1,...,m,\ j=0,1,...,m-1,$$

$$(F4)_{ij}=f(s_i,x_j)\quad i,j=1,...,m.$$

3. Stochastic Operational Matrix of TFs

Stochastic operational matrix of TFs to the Itô integral is derived in this section. We compute $\int_0^x T1_i(s)dB(s)$ and $\int_0^x T2_i(s)dB(s)$ as follows:

$$\int_0^x T1_i(s)dB(s)=\int_0^x\{u(s-ih)-\frac{1}{h}(s-ih)u(s-ih)+\frac{1}{h}(s-(i+1)h)u(s-(i+1)h)\}dB(s), \qquad (5)$$

and

$$\int_0^x T2_i(s)dB(s)=\int_0^x\{\frac{1}{h}(s-ih)u(s-ih)-\frac{1}{h}(s-(i+1)h)u(s-(i+1)h)-u(s-(i+1)h)\}dB(s), \qquad (6)$$

where, $u(x)$ is the unit step function. These integrations can be divided into tree cases. At first consider $x\in[0,ih)$:

$$\int_0^x T1_i(s)dB(s)=0, \qquad (7)$$

and

$$\int_0^x T2_i(s)dB(s)=0. \qquad (8)$$

For $x\in[ih,(i+1)h),$ we get:

$$\int_0^x T1_i(s)dB(s)=(i+1)\int_{ih}^x dB(s)-\frac{1}{h}\int_{ih}^x sdB(s) \qquad (9)$$

$$=(i+1)[B(x)-B(ih)]-\frac{1}{h}\int_{ih}^x sdB(s),$$

and

$$\int_0^x T2_i(s)dB(s) = \frac{1}{h}\int_{ih}^x sdB(s) - i\int_{ih}^x dB(s) \tag{10}$$

$$= \frac{1}{h}\int_{ih}^x sdB(s) - i[B(x) - B(ih)].$$

Finally, for the case of $x \in [(i+1)h, T)$, we get

$$\int_0^x T1_i(s)dB(s) = (i+1)\int_{ih}^{(i+1)h} dB(s) - \frac{1}{h}\int_{ih}^{(i+1)h} sdB(s) \tag{11}$$

$$= (i+1)[B((i+1)h) - B(ih)] - \frac{1}{h}\int_{ih}^{(i+1)h} sdB(s)$$

$$= \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds,$$

and

$$\int_0^x T2_i(s)dB(s) = \frac{1}{h}\int_{ih}^{(i+1)h} sdB(s) - i\int_{ih}^{(i+1)h} dB(s)$$

$$= \frac{1}{h}\int_{ih}^{(i+1)h} sdB(s) - i[B((i+1)h) - B(ih)]$$

$$= B((i+1)h) - \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds. \tag{12}$$

The result of these tree cases can be expanded in to TF series:

$$I(T1_i(x)) = \int_0^x T1_i(s)dB(s) \simeq [\xi_{i0},...,\xi_{im-1}]T1(x) + [\zeta_{i0},...,\zeta_{im-1}]T2(x), \tag{13}$$

and

$$I(T2_i(x)) = \int_0^x T2_i(s)dB(s) \simeq [\alpha_{i0},...,\alpha_{im-1}]T1(x) + [\beta_{i0},...,\beta_{im-1}]T2(x), \tag{14}$$

where, $\xi_{ij} = I(T1_i(jh)), \alpha_{ij} = I(T2_i(jh))$ and $\zeta_{ij} = I(T1_i((j+1)h)), \beta_j = I(T2_i((j+1)h))$ for $j = 0, 1, ..., m-1$. From Eqs.(7-12) we get

$$\xi_{ij} = \alpha_{ij} = 0, \quad j \le i,$$

$$\xi_{ij} = \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds, \ i < j,$$

$$\alpha_{ij} = B((i+1)h) - \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds, \ i < j,$$

and for $i = 0,...,m-1, \ j = 0,...,m-1, \zeta_{ij} = \xi_{ij+1}, \beta_{ij} = \alpha_{ij+1}$. Finally we can write

$$\int_0^x T1_i(s)dB(s) \simeq P1_s T1(x) + P2_s T2(x),$$

where, $P1_s$ and $P2_s$ are $m\times m$ stochastic operational matrices of TFs. These matrices can be obtained as follow:

$$P1_s = \begin{pmatrix} 0 & \xi_{01} & \xi_{02} & \dots & \xi_{0m-1} \\ 0 & 0 & \xi_{12} & \dots & \xi_{1m-1} \\ 0 & 0 & 0 & \dots & \xi_{2m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{m\times m},$$

and

$$P2_s = \begin{pmatrix} \xi_{01} & \xi_{02} & \xi_{03} & \dots & \xi_{0m} \\ 0 & \xi_{12} & \xi_{13} & \dots & \xi_{1m} \\ 0 & 0 & \xi_{23} & \dots & \xi_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \xi_{m-1m} \end{pmatrix}_{m\times m}.$$

In a similar manner, the Itô integration of T2(t) is

$$I(T2(x)) \simeq P3_s T1(x) + P4_s T2(x),$$

where,

$$P3_s = \begin{pmatrix} 0 & \alpha_{01} & \alpha_{02} & \dots & \alpha_{0m-1} \\ 0 & 0 & \alpha_{12} & \dots & \alpha_{1m-1} \\ 0 & 0 & 0 & \dots & \alpha_{2m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{m\times m},$$

and

$$P4_s = \begin{pmatrix} \alpha_{01} & \alpha_{02} & \alpha_{03} & \dots & \alpha_{0m} \\ 0 & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} \\ 0 & 0 & \alpha_{23} & \dots & \alpha_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{m-1m} \end{pmatrix}_{m\times m}.$$

Then we get

$$\int_0^x T(s)dB(s) = \begin{pmatrix} \int_0^x T1(s)dB(s) \\ \int_0^x T2(s)dB(s) \end{pmatrix} \simeq \begin{pmatrix} P1_s T1(x) + P2_s T2(x) \\ P3_s T1(x) + P4_s T2(x) \end{pmatrix} = \begin{pmatrix} P1_s & P2_s \\ P3_s & P4_s \end{pmatrix}\begin{pmatrix} T1(x) \\ T2(x) \end{pmatrix},$$

so,

$$\int_0^x T(s)dB(s) \simeq P_s T(x),$$

where, P_s, stochastic operational matrix of T(x), is

$$\begin{pmatrix} P1_s & P2_s \\ P3_s & P4_s \\ & \end{pmatrix}$$

The Itô integration of $\varphi(x)$ can be approximated as

$$\int_0^x \phi(s)dB(s) \simeq \Phi^T P_s T(x). \tag{15}$$

4. Solving N-dimensional Stochastic Integral Equation

Approximations of $y(x)$ $y_0, f(x,s)$ $g_j(x,s)$ $j=1,...,n,$ in TFs domain can be written as:

$$y_0 \simeq Y_0^T T(x) = T^T(x) Y_0, \tag{16}$$

$$y(x) \simeq Y^T T(x) = T^T(x) Y, \tag{17}$$

$$f(x,s) \simeq T^T(x) F T(s), \tag{18}$$

$$g_j(x,s) \simeq T^T(x) G_j T(s) \quad j=1,...,n, \tag{19}$$

such that 2m-vectors Y_0, Y, are stochastic TF coefficient, and $2m \times 2m$ matrices F and $G_j, j=1,...,n$ are TFs coefficients matrices. By substituting Eqs.(16-19) in (1) we get

$$Y^T T(x) \simeq Y_0^T T(x) + T^T(x) A (\int_0^x T(s) T^T(s) Y ds) + \sum_{j=1}^n T^T(x) G_j (\int_0^x T(s) T^T(s) Y dB_j)$$

By using (3) we can write

$$Y^T T(x) \simeq X_0^T T(x) + T^T(x) A (\int_0^x \tilde{Y} T(s) ds) + \sum_{j=1}^n T^T(x) G_j (\int_0^x \tilde{Y} T(s) dB_j),$$

or

$$Y^T T(x) \simeq Y_0^T T(x) + T^T(x) A \tilde{Y} P T(x) + \sum_{j=1}^n T^T(x) G_j \tilde{Y} P_s T(x),$$

Finally, by using (4) we get

$$Y^T T(x) \simeq Y_0^T T(x) + \hat{A}^T T(x) + \sum_{j=1}^n \hat{A}_j{}^T T(x)$$

where, $\hat{A}$ and $\hat{A}_j, j=1,...,n$ are $2m$ vectors with elements equal to the diagonal entries of $A\tilde{Y}P$ and $G_j \tilde{Y} P_s$, $j=1,...,n,$ respectively.

$$Y \simeq Y_0 + \hat{A} + \sum_{j=1}^n \hat{A}_j. \tag{20}$$

The linear system of equations in (20) can be solved easily.

5. Convergence Analysis

This section prepares convergence analysis of presented approach in $(C[0,1], \|.\|)$, continous functions in Banach space, with norm $\|\varphi(x)\| = max_{0 \le x \le 1} |\varphi(x)|$. The following error holds for all $\varphi \in L^2([0,1])$ that is expanded in TFs series[12]:

$$\| \phi(x) - \hat{\phi}(x) \| \le ch, \tag{21}$$

where, $\varphi(x)$ is defined in (2).

Theorem 5.1 Let $x(t)$ and $x_m(t)$ be the exact solution and approximate solution of (1) respectively and

i) $E|y(x)| \le D, \ x \in I = [0,1)$,

ii) $|f(x,s)| \le M$, $\quad |g_j(x,s)| \le M_j, (x,s) \in I \times I$, $j=1,...,n,$

hold then,

$$E\|y(x) - y_m(x)\| \to 0$$

Proof. Let $e_i(x) = y(x) - y_m(x)$ be the error function of approximate solution $x_m(t)$ to the exact solution $y(x)$ we can write

$$\|e_m(x)\| \le \|I_1\| + \|I_2\|, \tag{22}$$

where,

$$I_1 = \int_0^x [f(x,s)y(s) - \hat{f}(x,s)y_m(s)]ds,$$

$$I_2 = \sum_{j=1}^n \int_0^x [g_j(x,s)y(s) - \hat{g}_j(x,s)y_m(s)]dB_j(s).$$

For I_1 we get

$$E\|I_1\| \le \int_0^x E(\|f(x,s)\| \| \ y(s) - y_m(s)\|)ds + \int_0^x E(\|y_m(s)\| \|f(x,s) - \hat{f}(x,s)\|)ds,$$

$$\le M \int_0^x E\|e_m(s)\| ds + ch(\int_0^x E\|y(s) - y_m(s)\| ds + \int_0^x E\|y(s)\| ds)$$

$$\le M(1+ch) E \int_0^x \|e_m(s)\| ds + O(h), \tag{23}$$

Similarly, for I_2

$$E\|I_2\| \le E\| \sum_{j=1}^n \int_0^x [g_j(x,s)y(s) - \hat{g}_j(x,s)y_m(s)]dB_j(s)\|$$

$$\le \sum_{j=1}^n \int_0^x E\|g_j(x,s)y(s) - \hat{g}_j(x,s)y_m(s)\| ds$$

$$\le \sum_{j=1}^n \int_0^x E(\|g_j(x,s)\| \|y(s) - y_m(s)\|)ds + \int_0^x E(\|y_m(s)\| \|g_j(x,s) - \hat{g}_j(x,s)\|)ds,$$

$$\le \sum_{j=1}^n (M_j \int_0^x E\|e_m(s)\| ds + c_j h(\int_0^x E\|y(s) - y_m(s)\| ds + \int_0^x E\|y(s)\| ds))$$

$$\leq \sum_{j=1}^{n} M_j (1+c_j h) E\int_0^x \| e_m(s) \| ds + O(h). \quad (24)$$

From (22), (23) and (24) we conclude

$$E \| e_m(x) \| \leq \alpha \int_0^x E \| e_m(s) \| ds + O(h), \quad (25)$$

where, $\alpha = M(1+ch) + \sum_{j=1}^{n} M_j (1+c_j h)$. Gronwall inequality and (25) coclude

$$E \| e_m(x) \| \leq O(h)(1+\alpha \int_0^x e^{\alpha(x-s)} ds), \quad x \in [0,1),$$

By substituting $h = \frac{1}{m}$, and increasing m, it implies $\| e_m(x) \| \to 0$ as $m \to \infty$.

6. Numerical Examples

This section is devoted for solving some exampeles to show efficacy of presented approach.

Example 1. A linear stochastic integral equation is consideredas follows[10]

$$y(x) = y_0 + r\int_0^x y(s)ds + \sum_{j=1}^{n} \int_0^x \alpha_j y(s) dB_j(s), \quad x \in [0,1), \quad (26)$$

with the exact solution

$$y(x) = y_0 e^{(r - \frac{1}{2}\Sigma_{j=1}^{n} \alpha_j^2)x + \Sigma_{j=1}^{n} \alpha_j B(x)},$$

The numerical results for $y_0 = \frac{1}{200}, r = \frac{1}{20}, \alpha_1 = \frac{1}{50}, \alpha_2 = \frac{2}{50}, \alpha_3 = \frac{4}{50}, \alpha_4 = \frac{9}{50}$ are shown in Table 1. $\overline{x}_E$ is the errors mean and s_E is the standard deviation of errors in k iteration. In addition, we consider $y_0 = 0.5, \lambda = 1, \sigma = 0.25.$

Example 2. Consider following example[10]:

$$dy(x) = r(x)y(x)dx + \sum_{j=1}^{n} \sigma_j(x) y(x) dB(x), \quad x \in [0,1), \quad (27)$$

with the exact solution $y(x) = y_0 e^{\int_0^x (r(s) - \frac{1}{2}\sum_{j=1}^{n} \sigma_j^2(s))ds + \sum_{j=1}^{n} \int_0^x \sigma_j(s) dB_j(s)}$.

The numerical results for $x_0 = \frac{1}{12}, r(s) = s^2, \sigma_1(s) = sin(s), \sigma_2(s) = cos(s), \sigma_3(s) = s$ are shown in Table 2.

Table 1. xE(Mean) and sE(standard deviation)for k = 500

m	xE	sE
2^4	1.8×10^{-4}	1.5×10^{-4}
2^5	4.1×10^{-4}	2.9×10^{-4}
2^6	7.7×10^{-4}	4.1×10^{-4}
2^7	6.8×10^{-4}	5.6×10^{-4}

Table 2. xE(Mean) and sE(standard deviation)for k = 500(iteration)

m	xE	sE
2^4	1.7×10^{-3}	1.9×10^{-3}
2^6	4.3×10^{-4}	5.2×10^{-4}
2^5	5.8×10^{-4}	4.9×10^{-4}
2^7	2.8×10^{-4}	1.4×10^{-4}

7. Conclusion

In presented approach we obtained operational matrices of TFs to solve N-SDE. The properties of the TFs are used to convert the N-SDE to a system of linear algebraic equations. This presented approach reduces cost of computations due to properties of TFs. Also this approachis applied easily to solve N-SDE. Presented examples show good accuracy of this approach.

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