Indian Journal of Science and Technology, *Vol 10(1), DOI: 10.17485/ijst/2016/v10i1/109929, January 2017*
ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645

Quadrature Formula Study for the Integral with Hilbert Kernel based on Trigonometric Interpolational Polynomial

Almaz Ferdinantovich Gilemzyanov*, Anis Fuatovich Galimyanov and Chulpan Bakievna Minnegalieva

Kazan Federal University, 420008, Kazan, Kremlevskaya str. 18, Russia; Almaz.fe.gil@gmail.com, Anis.Fu.Ga12@yahoo.com, Ch.Bakievana.Ml@yahoo.com

Abstract

Objectives: The article presents the study of the quadrature formula for the integral with Hilbert's kernel. **Methods**: The sub integral function is close to interpolation polynomial on such equally spaced nodes that the values of the Weyl fractional integral coincide in these nodes for the function and polynomial. At the derivation of a formula, the known values of the integral are used with Hilbert's kernel of certain functions, the properties of trigonometric polynomials and the properties of trigonometric functions **Results**: The obtained quadrature formulas were tested using Wolfram Mathematica system. Calculations performed at different values of node number and the order of integration. The values obtained using the studied quadrature was compared with the values obtained using the previously known formula. **Conclusion**: The growth of node number improves by the quadrature formula, the dependence of approximation on the values, is observed. At the resemblance to the section ends the difference between integral values calculated by different formulas increases.

Keywords: Fractional Integration, Integral with the Hilbert Kernel, Interpolational Polynomial, Polynomial Operator, Quadrature Formula, Trigonometric Polynomial, Weyl Fractional Integral

1. Introduction

The issues of fractional integration and differentiation are being studied currently by various authors. In[1] provides a new definition of fractional derivative and fractional integral. In the work[2] the fractional derivative is regarded as the generalization of derivatives and integrals of the whole order. The solutions of fractional integral equations[3] are also studied. In[4] a new approach to fractional integration and the summary fractional Riemann-Liouville integral introduced, and some properties are proved.

Earlier in the work[5], we determined the form of a polynomial operator associating trigonometric polynomial with 2π -periodic function, satisfying the following conditions $I_{\pm}^{\alpha}(T_n)(t_k) = I_{\pm}^{\alpha}(\varphi)(t_k)$, where t_k t_k are equidistant nodes at (-π, π), , $I_{\pm}^{\alpha}$ – Weyl fractional integral.

**Author for correspondence*

In this paper, we study the quadrature formula for the integral with Hilbert kernel based on a trigonometric interpolational polynomial.

The integral similar to

$$S(\phi;t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(x) ctg\frac{t-x}{2}dx$$ is called the

Hilbert's integral, where $ctg\dfrac{t-x}{2}$ is Hilbert kernel.

This function has the peculiarity of the first order, therefore, an integral will not exist as a nonintrinsic one. This integral is understood in the sense of Cauchy principal value, i.e.,

$$S(\varphi;t) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(x)ctg\frac{t-x}{2}dx = \lim_{\varepsilon\to 0}\left[\frac{1}{2\pi}\int_0^{1-\varepsilon}\varphi(x)ctg\frac{t-x}{2}dx + \frac{1}{2\pi}\int_{1+\varepsilon}^{2\pi}\varphi(x)ctg\frac{t-x}{2}dx\right]$$

It is known that $S(e^{ikx};t) = i\,\mathrm{sgn}\,k \cdot e^{ikt}$,

$$\operatorname{sgn} k = \begin{cases} 1, k > 0 \\ 0, k = 0 \\ -1, k < 0 \end{cases}$$

$$S(\cos kx; t) = -\sin kt\,,\ k+1 \in N\,;$$

$$S(\sin kx; t) = \cos kt\,,\ k \in N\,;$$

$$S(const) = 0\,.$$

Quadrature process for a singular integral with the kernel of Hilbert type was proposed previously in[6], the methods of error estimates were also considered there. Radial based Kernel functions developed based on kernel of Hilbert is the core of support vector mapping for higher dimension representation[7–9].

2. Quadrature Formula Derivation for the Integral with Hilbert Kernel

Let's draw up quadrature formula for the integral with Hilbert kernel $S(I^{\alpha}\varphi; t)$.

Let's approximate the desired function by trigonometric polynomial[10]:

$$\phi(t) \approx \phi_n(t) = \frac{a_0}{2} + \sum_{k=-n}^{n} a_k \cos kt + b_k \sin kt\,,$$

where

$$a_k = \frac{2}{2n+1}\sum_{j=-n}^{n} \phi(t_j)\cos kt_j\,,$$

$$b_k = \frac{2}{2n+1}\sum_{j=-n}^{n} \phi(t_j)\sin kt_j \qquad (1)$$

Let's consider equally spaced nodes $t_j = \dfrac{2j\pi}{2n+1}$, $j = \overline{-n, n}$.

Then $I^{\alpha}(\varphi_n; t) =$

$$= \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \frac{a_0}{2}\Psi^{\alpha}(x)dx}_{=0} + \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=1}^{n}[a_k \cos k(t-x) + b_k \sin k(t-x)]\Psi^{\alpha}(x)dx =$$

$$\sum_{k=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}[a_k \cos k(t-x) + b_k \sin k(t-x)]\Psi^{\alpha}(x)dx =$$

$$\sum_{k=1}^{n}\underbrace{\frac{1}{2\pi}\int_0^{2\pi} a_k \cos k(t-x)\Psi^{\alpha}(x)dx}_{=I_1} + \sum_{k=1}^{n}\underbrace{\frac{1}{2\pi}\int_0^{2\pi} b_k \sin k(t-x)\Psi^{\alpha}(x)dx}_{=I_2}.$$

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} a_k \cos k(t-x)\Psi^{\alpha}(x)dx = \left(I^{\alpha} a_k \cos kx\right)(t) = a_k k^{-\alpha}\cos\left(kt - \frac{\alpha\pi}{2}\right).$$

Similarly

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi} b_k \sin k(t-x)\Psi^{\alpha}(x)dx = b_k k^{-\alpha}\sin\left(kt - \frac{\alpha\pi}{2}\right).$$

Therefore

$$I^{\alpha}(\phi_n; t) = \sum_{k=1}^{n} k^{-\alpha}\left[a_k \cos\left(kt - \frac{\alpha\pi}{2}\right) + b_k \sin\left(kt - \frac{\alpha\pi}{2}\right)\right] \quad (2).$$

Let's consider

$$S\left(I^{\alpha}\phi_n; t\right) = \frac{1}{2\pi}\int_0^{2\pi}\left[I^{\alpha}\phi_n(x)\right] ctg\frac{x-t}{2}dx =$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=1}^{n} k^{-\alpha}\left[a_k \cos\left(kt - \frac{\alpha\pi}{2}\right) + b_k \sin\left(kt - \frac{\alpha\pi}{2}\right)\right] ctg\frac{x-t}{2}dx =$$

$$\sum_{k=1}^{n} k^{-\alpha}\left[\int_0^{2\pi} a_k \cos\left(kt - \frac{\alpha\pi}{2}\right) ctg\frac{x-t}{2}dx + \int_0^{2\pi} b_k \text{ sir}\right.$$

$$\sum_{k=1}^{n} k^{-\alpha}\left[-a_k \sin\left(kt - \frac{\alpha\pi}{2}\right) + b_k \cos\left(kt - \frac{\alpha\pi}{2}\right)\right]. \quad (3)$$

Let's substitute a_k, b_k by Formula (1) and consider Formula (2) and (3):

$$I^{\alpha}(\varphi_n; t) =$$

$$= \sum_{k=1}^{n} k^{-\alpha}\left[\frac{2}{2n+1}\sum_{j=-n}^{n}\phi(t_j)\cos kt_j \cdot \cos\left(kt - \frac{\alpha\pi}{2}\right) + \frac{2}{2n+1}\sum_{j=-n}^{n}\phi(t_j)\sin kt_j \cdot \sin\left(kt - \frac{\alpha\pi}{2}\right)\right] =$$

$$\frac{2}{2n+1}\sum_{j=-n}^{n}\phi(t_j)\sum_{k=1}^{n} k^{-\alpha}\left[\cos kt_j \cdot \cos\left(kt - \frac{\alpha\pi}{2}\right) + \sin kt_j \cdot \sin\left(kt - \frac{\alpha\pi}{2}\right)\right] =$$

$$\frac{2}{2n+1}\sum_{j=-n}^{n}\phi(t_j)\sum_{k=1}^{n} k^{-\alpha}\cos\left[k(t - t_j) - \frac{\alpha\pi}{2}\right].$$

Let's take into account that

$$\Psi_{-}^{\alpha}(t) = 2\sum_{k=1}^{\infty}\frac{\cos\left(kt + \frac{\alpha\pi}{2}\right)}{k^{\alpha}}\text{, then}$$

$$I^{\alpha}(\phi_n; t) = \frac{1}{2n+1}\sum_{j=-n}^{n}\phi(t_j)\Psi_{n-}^{\alpha}(t_j - t)\text{, where}$$

$$\Psi_{n\pm}^{\alpha}(t) = 2\sum_{k=1}^{n}\frac{\cos\left(kt \mp \frac{\alpha\pi}{2}\right)}{k^{\alpha}}.$$

$$S(I^{\alpha}\varphi_n;t) =$$

$$=\sum_{k=1}^{n} k^{-\alpha}\left[-\frac{2}{2n+1}\sum_{j=-n}^{n}\phi(t_j)\cos kt_j\cdot\sin\left(kt-\frac{\alpha\pi}{2}\right)+\frac{2}{2n+1}\sum_{j=-n}^{n}\phi(t_j)\sin kt_j\cdot\cos\left(kt-\frac{\alpha\pi}{2}\right)\right]=$$

$$\frac{2}{2n+1}\sum_{j=-n}^{n}\phi(t_j)\sum_{k=1}^{n}k^{-\alpha}\sin\left(k(t_j-t)+\frac{\alpha\pi}{2}\right).$$

Let's denote via .

$$\overline{\Psi}_{n\pm}^{\alpha}(t)=2\sum_{k=1}^{n}\frac{\sin\left(kt\mp\frac{\alpha\pi}{2}\right)}{k^{\alpha}}$$

In order to prove the existence of $S(I^{\alpha}\varphi_n;t)$, it is necessary that the series $\overline{\Psi}_{n\pm}^{\alpha}(t)$ coincides.

Otherwise the series $\overline{\Psi}_{n\pm}^{\alpha}(t)$ may be presented as the following one:

$$\overline{\Psi}_{n\pm}^{\alpha}(t)=2\cos\frac{\alpha\pi}{2}\sum_{k=1}^{n}\frac{\sin kt}{k^{\alpha}}\mp 2\sin\frac{\alpha\pi}{2}\sum_{k=1}^{n}\frac{\cos kt}{k^{\alpha}}.$$

The convergence of such a series at $t\neq 0$ follows from the theorem

Theorem

If the series $u_0(x)+u_1(x)+\ldots+u_n(x)+\ldots$ converges uniformly and if $\{w_v\}$ there is a sequence of bounded variation, then the series $u_0(x)w_0+u_1(x)w_1+\ldots$ also converges uniformly. If the partial sums of the series $u_0(x)+u_1(x)+\ldots+u_n(x)+\ldots$ are limited uniformly and if the sequence $\{w_v\}$ with a limited change is close to zero, then the series $u_0(x)w_0+u_1(x)w_1+\ldots$ converges uniformly.

Then

$$S(I^{\alpha}\phi_n;t)=\frac{1}{2n+1}\sum_{j=-n}^{n}\phi(t_j)\overline{\Psi}_{n-}^{\alpha}(t_j-t).$$

Quadrature formula for the integral with Hilbert kernel based on $A_n(\varphi;t)$

Let us consider the singular integral with the Hilbert kernel

$$S(\phi;t)=\frac{1}{2\pi}\int_0^{2\pi}\phi(x)ctg\frac{t-x}{2}dx$$

Let's approximate a sub integral function $\varphi(x)$ by interpolational polynomial

$$\phi(t)\approx A_n(\phi;t)=\frac{1}{2n+1}\sum_{j=-n}^{n}I^{\alpha}(\phi;t_j)\sum_{|k|=1}^{n}(ik)^{\alpha}e^{i[k(t-t_j)]}.$$

$$S(\phi;t)=\frac{1}{2\pi}\int_0^{2\pi}\phi(x)ctg\frac{t-x}{2}dx\approx\frac{1}{2\pi}\int_0^{2\pi}A_n(\phi;x)ctg\frac{t-x}{2}dx=$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2n+1}\sum_{j=-n}^{n}I^{\alpha}(\phi;t_j)\sum_{|k|=1}^{n}(ik)^{\alpha}e^{i[k(x-t_j)]}ctg\frac{t-x}{2}dx=$$

$$=\frac{1}{2n+1}\sum_{j=-n}^{n}I^{\alpha}(\phi;t_j)\sum_{|k|=1}^{n}(ik)^{\alpha}e^{-ikt_j}\frac{1}{2\pi}\int_0^{2\pi}e^{ikx}ctg\frac{t-x}{2}dx$$

It is known that

$$\frac{1}{2\pi}\int_0^{2\pi}e^{ikx}ctg\frac{t-x}{2}dx=i\,\mathrm{sgn}\,k e^{ikt}.$$

Then we get the following formula:

$$S(\phi;t)\approx\frac{i}{2n+1}\sum_{j=-n}^{n}I^{\alpha}(\phi;t_j)\sum_{|k|=1}^{n}\mathrm{sgn}\,k(ik)^{\alpha}e^{ik(t-t_j)}.$$

Let's use that $(ik)^{\alpha}=|k|^{\alpha}e^{\frac{\alpha\pi i}{2}\mathrm{sgn}\,k}$ 11.

Let's $t-t_j=p$

$$\sum_{|k|=1}^{n}\mathrm{sgn}\,k(ik)^{\alpha}e^{ikp}=\sum_{|k|=1}^{n}\mathrm{sgn}\,k|k|^{\alpha}e^{\frac{\alpha\pi i}{2}\mathrm{sgn}\,k+ikp}=\sum_{|k|=1}^{n}\mathrm{sgn}\,k|k|^{\alpha}e^{i[\frac{\alpha\pi}{2}\mathrm{sgn}\,k+kp]}=$$

$$\sum_{|k|=1}^{n}\mathrm{sgn}\,k|k|^{\alpha}e^{i[\frac{\alpha\pi}{2}\mathrm{sgn}\,k+kp]}=\sum_{|k|=1}^{n}\mathrm{sgn}\,k|k|^{\alpha}\left[\cos\left(\frac{\alpha\pi}{2}\mathrm{sgn}\,k+kp\right)+i\sin\left(\frac{\alpha\pi}{2}\mathrm{sgn}\,k+kp\right)\right]=$$

$$=\sum_{|k|=1}^{n}\mathrm{sgn}\,k|k|^{\alpha}\left[\cos\left(\frac{\alpha\pi}{2}\mathrm{sgn}\,k\right)\cdot\cos kp-\sin\left(\frac{\alpha\pi}{2}\mathrm{sgn}\,k\right)\cdot\sin kp+\right.$$

$$\left.+i\sin\left(\frac{\alpha\pi}{2}\mathrm{sgn}\,k\right)\cdot\cos kp+i\cos\left(\frac{\alpha\pi}{2}\mathrm{sgn}\,k\right)\cdot\sin kp\right]=$$

$$=\sum_{k=1}^{n}k^{\alpha}\left[\cos\left(\frac{\alpha\pi}{2}\right)\cdot\cos kp-\sin\left(\frac{\alpha\pi}{2}\right)\cdot\sin kp+i\sin\left(\frac{\alpha\pi}{2}\right)\cdot\cos kp+i\cos\left(\frac{\alpha\pi}{2}\right)\cdot\sin kp-\right.$$

$$\left.-\cos\left(-\frac{\alpha\pi}{2}\right)\cdot\cos(-kp)+\sin\left(-\frac{\alpha\pi}{2}\right)\cdot\sin(-kp)-i\sin\left(-\frac{\alpha\pi}{2}\right)\cdot\cos(-kp)-i\cos\left(-\frac{\alpha\pi}{2}\right)\cdot\sin(-kp)\right]=$$

$$=\sum_{k=1}^{n}k^{\alpha}\left[\cos\left(\frac{\alpha\pi}{2}\right)\cdot\cos kp-\sin\left(\frac{\alpha\pi}{2}\right)\cdot\sin kp+i\sin\left(\frac{\alpha\pi}{2}\right)\cdot\cos kp+i\cos\left(\frac{\alpha\pi}{2}\right)\cdot\sin kp-\right.$$

$$\left.-\cos\left(\frac{\alpha\pi}{2}\right)\cdot\cos kp+\sin\left(\frac{\alpha\pi}{2}\right)\cdot\sin kp+i\sin\left(\frac{\alpha\pi}{2}\right)\cdot\cos kp+i\cos\left(\frac{\alpha\pi}{2}\right)\cdot\sin kp\right]=$$

$$=\sum_{k=1}^{n}k^{\alpha}\left[2i\sin\left(\frac{\alpha\pi}{2}\right)\cdot\cos kp+2i\cos\left(\frac{\alpha\pi}{2}\right)\cdot\sin kp\right]=2i\sum_{k=1}^{n}k^{\alpha}\cdot\sin\left(\frac{\alpha\pi}{2}+kp\right).$$

Substituting this result, we obtain the quadrature formula for the integral with Hilbert kernel:

$$\frac{1}{2\pi}\int_0^{2\pi}\phi(x)ctg\frac{t-x}{2}dx=\frac{-2}{2n+1}\sum_{j=-n}^{n}I^\alpha(\phi;t_j)\sum_{k=1}^{n}k^\alpha\sin\left(\frac{\alpha\pi}{2}+k(t-t_j)\right) \quad (4).$$

widely adopted the mentioned results to apply it on many different mechanical problems[12–17].

3. Calculations in Wolfram Mathematica System

Here are the results of calculations carried out in Wolfram Mathematica system. This method is often used in calculations[18].

As for the studied case Weyl fractional integral coincides with the fractional Riemann-Liouville integral, we compared at first the values of the fractional integral for the function $\varphi(t)$=sin3, 6t at n=6, α=0,6 calculated according to the formula

$$I_+^\alpha\phi=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}\frac{\varphi(x)dx}{(t-x)^{1-\alpha}}$$

and calculated according to the formula

$$I^\alpha(\phi_n;t)=\sum_{k=1}^{n}k^{-\alpha}\left[a_k\cos\left(kt-\frac{\alpha\pi}{2}\right)+b_k\sin\left(kt-\frac{\alpha\pi}{2}\right)\right. \quad (2)$$

The results of calculations for $\varphi(t)$=sin3, 6t at n=20, α=0,6 are shown on Figure 1:

Then we checked the obtained Formula (4) for the integral with the Hilbert kernel:

$$\frac{1}{2\pi}\int_0^{2\pi}\phi(x)ctg\frac{t-x}{2}dx=\frac{-2}{2n+1}\sum_{j=-n}^{n}I^\alpha(\phi;t_j)\sum_{k=1}^{n}k^\alpha\sin\left(\frac{\alpha\pi}{2}+k(t-t_j)\right) \quad (4)$$

In order to calculate the integral with the Hilbert kernel, we used the formula given in[19]:

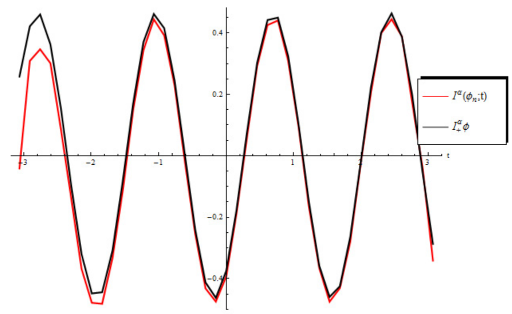

Figure 1. Approximated calculations.

$$\frac{1}{2\pi}\int_0^{2\pi}\phi(x)ctg\frac{t-x}{2}dx=\frac{2}{2n+1}\sum_{k=0}^{2n}\phi_k\frac{\sin(n+1)\frac{t_k-x}{2}\cdot\sin n\frac{t_k-x}{2}}{\sin\frac{t_k-x}{2}} \quad (5)$$

Figure 2 shows the values calculated according to the Formulas (4) and (5) for the function $\varphi(t)$=sin3, 6t at n=50, and α=0,6 on the section [0.1, 3] with the step of 0.01. In this case the values $I^\alpha(\varphi;t_j)$ are calculated according to the following formula

$$I_+^\alpha\phi=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}\frac{\varphi(x)dx}{(t-x)^{1-\alpha}}$$

Figure 3 shows the values calculated according to the Formulae (4) and (5) for the function $\varphi(t)$=sin3, 6t at n=100, and α=0,6 on the section [0.1, 3]. The values $I^\alpha(\varphi;t_j)$ are calculated according to the following formula:

Table 1. Compared at first the values of the fractional integral

t	$-\frac{12}{13\pi}$	$-\frac{10}{13\pi}$	$-\frac{8}{13\pi}$	$-\frac{6}{13\pi}$	$-\frac{4}{13\pi}$	$-\frac{2}{13\pi}$	0
$I_+^\alpha\varphi$	0,1887	0,12625	−0,54124	0,07313	0,39437	−0,18985	−0,41678
$I^\alpha(\varphi_n;t)$	0,42945	0,10006	−0,46314	0,05589	0,44432	−0,2055	−0,37513

t	$\frac{2}{13\pi}$	$\frac{4}{13\pi}$	$\frac{6}{13\pi}$	$\frac{8}{13\pi}$	$\frac{10}{13\pi}$	$\frac{12}{13\pi}$
$I_+^\alpha\varphi$	0,35012	0,22119	−0,39345	−0,17624	0,51682	−0,15303
$I^\alpha(\varphi_n;t)$	0,33181	0,2634	−0,4205	−0,12181	0,46152	−0,03359

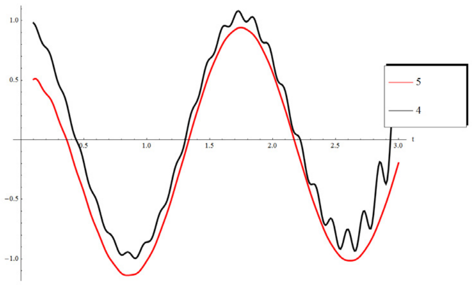

Figure 2. Integral values with Hilbert kernel, n=50, α =0,6.

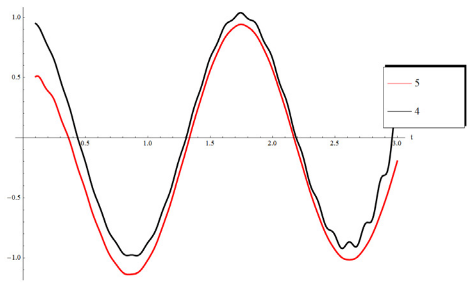

Figure 3. The values of the integral with the Hilbert kernel, calculated according to various formulas.

$$I^{\alpha}(\phi_n;t)=\sum_{k=1}^{n}k^{-\alpha}\left[a_k\cos\left(kt-\frac{\alpha\pi}{2}\right)+b_k\sin\left(kt-\frac{\alpha\pi}{2}\right)\right]$$

Figure 4 shows the values calculated according to the Formulas (4) and (5) for the function $\varphi(t)$=sin3, 6t at n=100, and α=0,6 on the section [0.1, 3].

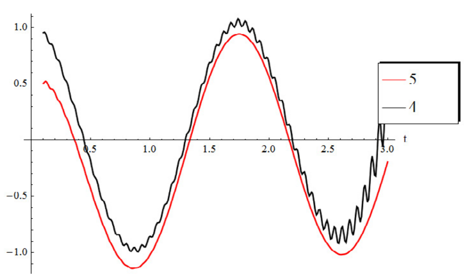

Figure 4. The values of the integral with the Hilbert kernel, n=100, α=0,6

Figure 5 shows the values calculated according to the Formulas (4) and (5) for the function $\varphi(t)$=sin3, 6t at n=100, and α=0,9 on the section [0.1, 3].

4. Conclusions

The values obtained using a studied quadrature formula compared with the values obtained using a previously known method. The growth of node number of n improves the approximation, the dependence of the estimate on values is observed.

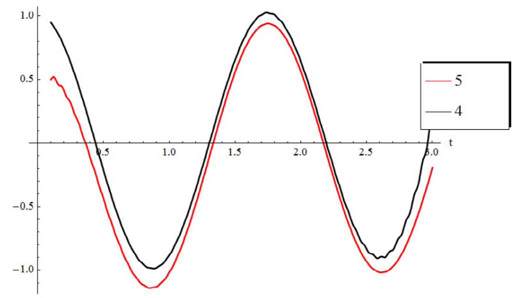

Figure 5. The values of the integral with the Hilbert kernel, n=100, α=0,9 .

5. Summary

Thus, quadrature formula for an integral with Hilbert kernel based on trigonometric interpolational polynomial are determined and studied. The check of the formula is performed using Wolfram Mathematica system. The work in the system showed that with the increasing number of nodes n the approximation is improved that is usually observed for similar tasks.

6. Conflict of Interest

The authors acknowledge that the presented data do not contain any conflict of interest.

7. Acknowledgements

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University.

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