A Review on Furstenberg Family in Dynamical Systems

Rakesh K. Pandey[1], Rashmi Singh[2] and Vinod Kumar[3]

[1]Department of Mathematics, JIL Information Technology, Sector-128, Noida, Uttar Pradesh, India;
rakesh_pandey1978@rediffmail.com
[2]Amity Institute of Applied Science, Amity University, Sector-125, Noida, Uttar Pradesh India; rsingh7@amity.edu
[3]Department of Mathematics, Kuruskhetra University, Kurukshetra, Haryana, India; vkvinod96@yahoo.co.in

Abstract

Objective: The purpose of this paper to highlight the comprehensive overview of Furstenberg family which creates a bridge for future research. **Methods:** For this, a detailed study of topology as well as dynamical systems is required. In addition to this a deep study of the term chaos as well as Furstenberg family is required. Also, the main of application of dynamical system is chaos theories which play an important role in the study of Furstenberg Family using topology. **Findings:** This paper demonstrates the development of dynamical system and some important terms related to dynamical systems such as topological entropy, weakly mixing sets, topologically transitive, period, orbit etc are well explained. The beauty of this study lies in its overview on the term chaos as well as its role in the study of dynamical system using topology. This paper brings a great attention on the term Furstenberg Family which is studied on the platform made by the combination of dynamical system and topology. **Application:** This paper gives a fruitful overview on Furstenberg family in topological dynamical systems having chaotic nature.

Keywords: Chaos Mathematics Subject Classification: 37D45, 37N05, 37C25, 54H20, Dynamical Systems, Furstenberg Family, Sensitivity, Transitivity

1. Introduction

The exploration of enduring behaviour of developing systems gives the birth of dynamical systems. Henri Poincare was the first pioneer who had explored many hypothesis associated with dynamical systems during his study on celestial mechanics. In 1961, S. Smale's discovered structurally stable dynamical systems having infinite periodic orbits. Introduced the term topological entropy[1], while[2] Introduce the concept of weak mixing[3]. Had coined the term chaos. After this, the term sensitivity was introduced[4]. Based on this term[5] had proposed the idea of sensitive dependence[6] on preliminary state of neighbourhood points of the orbits showing exponential divergence. Further in 1989 people got a new name of chaos as given by[7]. Afterward[8] found that Li-Yorke chaos is due to weak mixing. Further[9] gave an evidence for the fact that weakly mixing systems have points whose orbits exhibit highly inconsistent time dependence. In 1994[10], had introduced the term Distributional chaos.

2. Preliminaries

2.1 Dynamical System

This section will give an introduction to dynamical systems.

Author for correspondence

Dynamical system is a collection of probable states governed by certain rules to determine the current state of the system in context of the past state. There are some models where current state is not governed by fixed rules. For example, the prices of commodities keep fluctuating; therefore in this instance, there is no specific rule. Hence, it is not the case of dynamical system even though the current state is governed by time[11]. Dynamical system is categorized mainly into two different forms. When the dynamical system is restricted by discrete time, it is termed as discrete-time system/dynamical system. In such system, input and output are the current state and new state respectively. State of the system is a rule which is governed by some information referred as input for the system[12]. Composition of mapping is an evolutionary process of studying dynamical system[5]. If $f : S \to S, S \neq \phi$, then by using mathematical induction, you can obtained the functions $f^1, f^2, f^3, \ldots, f^{n-1}, f^n \left(= f^{n-1} o f\right)$ using iteration of function f[13].

Dynamical systems have different notations for its categorised forms. The mapping $f : X \to X, X \neq \phi$ is termed as discrete dynamical system, where $f^{n+1} = f^n o f, \forall n \in N$ and $f^0 = I\left(\text{Identity function}\right)$. In dynamical system the composite mapping is defined as $f^{m+n} = f^m o f^n, \forall m, n \in \mathbb{N}$, while the invertible mapping for isomorphic function f is defined as $f^{-n} = \left(f^n\right)^{-1}, \forall n \in \mathbb{N}$. For the dynamical systems (X, f) and (Y, g) we get anew dynamical system $h(x, y) = \left(f(x), g(y)\right); x \in X, y \in Y$ and its inverse function is defined as $h^{-1}(x, y) = \left(f^{-1}(x), g^{-1}(y)\right); x \in X, y \in Y$.

The second main category of dynamical system is the limiting case of discrete dynamical systems where inputs are updated even for very small interval of time. For such system, the governing rule consists of sequence of differential equations; which referred as continuous-time dynamical system.

In dynamical system forecasting of developing system is performed with the passage of time. Thus, for given value of x, you will get the function $f\ (x)$ for large k.

In dynamical system (X, f), for any point $a \in X$, $a, f(a), f^2(a), f^3(a), \ldots, f^{n-1}(a), f^n(a)$ termed

as orbit of a and a is known as the seed or initial value of the orbit[11]. Since, here for the function f domain and range both are the same; hence it is termed as map. Some time the functional value at any stage becomes equal to its initial value. The initial value a is termed as fixed point of the map[5], i.e., $f(a) = a$ [13]. The iterated value of the function $p : X \to X, X \neq \phi$ defined by $p(x) = 2x(1-x)$ are given by {0.01, 0.0198, 0.0388,..} which represents the orbits of p. Here, $p(a) = a$ for $a = 0, \dfrac{1}{2}$, thus these are the fixed point of p. It is not necessary that every point of an orbit is a fixed point; there are some points on the map of the function which are not fixed and termed as eventually fixed. For example, x = −1 which is a seed of the function f defined by $f(x) = x^2$ is eventually fixed since $f(-1) = 1 \neq -1$ but $f^2(-1) = f\left(f(-1)\right) = f(1) = 1$. Thus, the point 1 lying on the orbit of x is fixed which shows that x is eventuallyfixed[13]. The fixed point of the orbit may be stable or unstable that can be checked using differentiation. There are functions whose derivatives of all the orders exist and are always continuous functions recall as smooth functions.

If $f : R \to R$ and a is any its fixed point. Then,

- a is of attracting nature if $ I = (\alpha, \beta)$ and a ÎI, if x is any member of the open interval I then $\lim\limits_{n \to \infty} f^n(x) = a$.

- a is of repelling nature if $ I = (\alpha, \beta)$ and a ÎI, if x is any member of the open interval I then $\lim\limits_{n \to \infty} f^n(x) \neq a$.

There are some points termed as neutral fixed point which is neither attracting nor repelling[13].

There is some point on a map whose value is equal to the functional value of some iterated function; this value is termed as periodic point. Thus, point α is known as a periodic point of period m or period-m point if for any smallest positive integer m such that $f^m(\alpha) = \alpha$ and the orbit associated with this point is recall as a periodic orbit of period m or period-m orbit.

For example, the function f defined by $f(x) = x^2 - 1$ has a periodic point $\alpha = 0$ of period m = 2, since $f^2(\alpha) = f\left(f(\alpha)\right) = f(-1) = 0 = \alpha$. It is not necessary that every seed α of the orbit becomes periodic point, there

are some points in the orbit of x_0 which becomes periodic point and this x_0 is said to be eventually periodic[13]. The next section will give attention on topological dynamical systems.

2.2 Topological Dynamical System

In 1927, the Russian scientist - G.D. Birkhoff had extended the research of Poincare on celestial mechanics and Hadamard's on geodesic flows and gave new name to dynamical system as topological dynamical systems in which qualitative and asymptotic properties of dynamical system is explored. The dynamical system is governed by the time factor[14]. Thus, the metric d and continuous function f defined over the non-empty set X constitute a dynamical system (X, f) [13]. The given system is said to be trivial if it contain single point, for such system mapping is said to be an identity mapping which is always unique[14]. If the metric d defined over a compact metric space X having a countable basis the given dynamical system (X, f) recall as topological dynamical system. A discrete dynamical system becomes a topological dynamical system if the continuous mapping f is one-one, onto and f^{-1} is continuous. For this case topological dynamical system is also invertible. If there is homeomorphism h between the metric spaces (X_1, d_1) and (X_2, d_2) respectively such that $f_2 \circ h = h \circ f_1$ i.e., $f_2(h(x)) = h(f_1(x)) \forall x \in X_1$, then the dynamical systems (X_1, f_1) and (X_2, f_2) defined over the given metric spaces are said to be conjugate dynamical systems. Here, the mapping h is called the conjugate map. If for any seed a of the system X, if its orbit $\{a, f(a), f^2(a), f^3(a), \ldots, f^{n-1}(a), f^n(a), \ldots\}$ is dense in X, then the dynamical system (X, f) is said to be transitive[13]. Also f is isometric if it preserves the distance[12]. The n-th fold product for $n \geq 2$ of the topological dynamical space (X, f) is denoted by (X^n, f^n). The diagonal of X^n is Δ_n where $\Delta_n = \{(x, x, \ldots, x, \ldots, x) \in X^n\}$ and $\Delta^{(n)} = \{(x_1, x_2, \ldots, x_n) \in X^n\}$, then $\exists 1 \leq i < j \leq n$ such that $x_i = x_j$. For each member x of (X, f) its orbits is the set denoted by $\text{Orb}(x, f) = \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ and its limiting case is $\omega(x, f) = \bigcap_{n=1}^{\infty} \overline{\{f^i(x), i \geq n\}}$. Every

member of the limiting case is recall as recurrent point[14] and its collection $R(f)$ is f-invariant. Periodic points are always recurrent. In (X, f) if f is invertible, then α-limit set of x is $\alpha(x) = \alpha(x, f) = \bigcap_{n \in N} \overline{\bigcup \{f^{-i}x, i \geq n\}}$. Both the sets $\omega(x, f)$ and $\alpha(x, f)$ are closed as well as f-invariant. A point x the member of the phase space X is non-wandering point if for any neighbourhood G of x $\exists n \in \mathbb{N}$ such that $f^n(G) \cap G = \phi$. The set of non-wandering points $NW(x, f)$ is closed, f -invariant, and contains the points of $\omega(x)$ and $\alpha(x)$ for all the members x of X $\Rightarrow R(f) \subset NW(x, f)$. If Y is a non-empty closed invariant subset of X, then (Y, f) is also a dynamical system recall as a subsystem of (X, f).

If $\phi \neq Y \subset X$ where $\overline{Y} = Y$ and $f(Y) \subset Y$, then Y is f-minimal subset of X. A compact invariant set Y is minimal iff $f^{n+i}(y) \subset Y \forall y \in Y, i \in \mathbb{Z}_+$ and $f^{n+i}(y)$ is dense in Y. Thus, a periodic orbit is a minimal set. Topological dynamical system is consistently minimal if and only if it does not have any proper subsystem. Thus, any non-empty closed invariant subset Y of X is minimal if $f^n(y_i) \cap Y \neq \phi \forall y_i \in Y, n \in \mathbb{N}$. The member x of the phase space X termed as minimal point or almost periodic point if it is the member of minimal set.

(X, f) is weakly mixing if $(X \times X, f \times f)$ is transitive[15]. The dynamical system is transitive and strongly mixing if $f^n(A) \cap B \neq \phi \forall \phi \neq A, B \text{(open sets)}, n \in \mathbb{N}$ [1,6]. (X, f) is totally transitive if (X, f^n) is transitive $\forall n \in N$. Thus, the transitive points are the points having dense orbit. Hence, $\text{Trans}(X, f)$ (set of all transitive points) is a dense G_δ subset of X[17]. Any member (α, β) of (X, f) behaves

- Asymptotically if $\lim_{n \to \infty} d(f^n(\alpha), f^n(\beta)) = 0$

- Proximally if $\liminf_{n \to \infty} d(f^n(\alpha), f^n(\beta)) = 0$

- Distally if $\liminf_{n \to \infty} d(f^n(\alpha), f^n(\beta)) > 0$

The set of all proximal pairs P exhibit reflexive, symmetric, f-invariant relation but generally P is neither transitive nor closed[18-20]. For $(x, y) \in P$, $P(x)$ represents proximal cell shows minimality. Also, every subset of Orb(x) having minimality coincides with proximal cell[18]. Also system having a weakly mixing property have proximal cell as residue[21]. In addition to this[22], explained this concept in detailed manner. Now if proximal cell and minimal point becomes equal then that point termed as distal point and the system containing these points is termed as distal system. Also, the system whose distal points have dense orbit then that system is recall as point distal system.

If $f^n(U_i) \cap V_i \neq \phi \ \forall \ \phi \neq U_i, V_i \ (\text{open sets}), i = 1, 2 \ \& \ n \in \mathbb{N}$ then f is weakly mixing[23]. Again, if $f^m(U) \cap V \neq \phi \ \forall \ m \geq n, m, n \in \mathbb{N} \ \& \ \phi \neq U, V \ (\text{open sets}) \subset X$ then the mapping $f : X \rightarrow X$ termed as topologically mixing[24]. Further, [1]initiate the term topological entropy in (X, f). If where $\mathcal{S}$ be the sub-cover of $\mathcal{O}$ with minimum cardinality denoted by $\mathcal{N}_{\mathcal{S}}$ then $\log \mathcal{N}_{\mathcal{S}} = H(\mathcal{S})$ represents the entropy related with $\mathcal{O}$. Thus, for the mapping $f : X \rightarrow X$ open cover of X is represented by $f^{-1}(\mathcal{S}) = \{ f^{-1}(G_i) : G_i \in \mathcal{S} \}$.

If $h_{f, \mathcal{S}} = \lim_{n \to \infty} \dfrac{H(\mathcal{S} \vee f^{-1}(\mathcal{S}) \vee f^{-2}(\mathcal{S}) \vee \ldots \vee f^{-n+1}(\mathcal{S}))}{n}$

then topological entropy of (X, f) is defined by $h(f) = \sup h_{f, \mathcal{S}}$[25]. Next section is an overview on chaotic dynamical system.

2.3 Chaotic Dynamical System

In recent years study of chaos (disorder) in dynamical systems becomes great interest of researchers. Chaos was initially coined in 1975 by two researchers[25] during the description of complex attributes of the trajectories of the orbit. Till now so many alternative definitions of chaos had been proposed but even after long time of study we still waiting for exact precise definition of chaos. Generally the different definition of chaos was depends on the following features:

- Points whose orbit is complex, having Li-Yorke and distributional chaos;
- Points having sensitivity dependence, having Devaney and Auslander-Yorke chaos also having Li-Yorke sensitivity;

- Distinct orbits whose length is n and have fast growth, have positive topological entropy;
- Points which possess powerful iteration, having weakly mixing sets and property.

The above features exhibit the common features exist in the definition of chaos. Some researchers had currently tried to associate the sensitivity conditions with the new definition of chaos given by Li-Yorke. In[26] had given a new name to chaos known as spatiotemporal chaos. According to them, a topological dynamical system (X, f) have spatiotemporally chaotic nature if each element a∈X is a limiting value of another member b∈X such that the ordered duple behaves proximally but not asymptotically. In other words (α, β) is termed as Li-Yorke scrambled pair[25] if they satisfy the following conditions $\liminf_{n \to \infty} d(f^n(\alpha), f^n(\beta)) = 0$ and $\limsup_{n \to \infty} d(f^n(\alpha), f^n(\beta)) > 0$. In[27] was the first person who had introduced the theory scrambled set. The distinct elements of scrambled pair (x, y) constitute scrambled set S. Chaotic nature of topological dynamical system is Li-Yorke if the scrambled set S of the given system is uncountable[14].

Further[2] had introduced the idea of Li-Yorke sensitivity. A system is Li-Yorke sensitive if for each limiting value a of another member b of X ∃ε > 0 such that the ordered duple (α, β) is proximal but not ε-asymptotic if

$\liminf_{n \to \infty} d(f^n(\alpha), f^n(\beta)) = 0$ and $\limsup_{n \to \infty} d(f^n(\alpha), f^n(\beta)) > \varepsilon$.

A following remark can be drawn from the given concepts.

Remark: EveryLi-Yorke sensitive system is being spatiotemporal chaotic but converse need not be true.
Some more results proposed[25].

Theorem 1. A continuous function mapping between two unit closed interval having periodic point of order-3 haveLi-Yorke chaos[25].

In topological dynamical system (X, f) the scrambled duple (α, β) is termed as -scrambled duple if for any positive real number , $\liminf_{n \to \infty} d(f^n(\alpha), f^n(\beta)) = 0$ and $\limsup_{n \to \infty} d(f^n(\alpha), f^n(\beta)) > \delta$.

Again the distinct member of the -scrambled duple constitute -scrambled set S′⊂X. Also, the topological dynamical system (X, f) is Li-Yorke -chaotic if S□ is uncountable[14]. For any positive real number the Li-Yorke chaotic system is Auslander-Floyd system[19] if it have no uncountable scrambled sets. In[28] had given evidence that continuous functions defined on unit intervals having positive topological entropy will have -scrambled set for > 0. In 1986,[29] and[30] independently proved that there are continuous map have Li-Yorke chaos whose topological entropy is zero.

Theorem 2. The topological entropy of a continuous function mapping between two unit closed interval having Li-Yorke chaos is zero[29,30].

Every system having weak mixing property have Li-Yorke chaos, this theory was proved by[8] with the help of Mycielski Theorem.

Theorem 3. For some positive real number the non-trivial dynamical systemhaving weak mixing property has dense Mycielski -scrambled subset of X[31].

Remarks:

- The dynamical system associated with the scrambled space is completely scrambled and hence proximal.
- It is not necessary that for any > 0, the whole space cannot be -scrambled.

Here, we have some characteristics of topological dynamical system behaves proximally.

Theorem 4. Topological dynamical system behaves proximally if it have unique minimal point[8,32].

In 1997,[33] proved that non-compact spaces have a few wholly scrambled system.

Theorem 5. If there is a uniform homomorphism between a metric space X and open cube $O^n = (0,1)^n \ \forall n \geq 2$. Then there would be homeomorphism between the entire spaces X constituting scrambled set X.

In 2001,[34] proved that there are compact spaces which allow few wholly scrambled systems.

Theorem 6. Compact metric space exhibiting completely scrambled homeomorphisms, have continuous countable compact metric space and the Cantor set of arbitrary dimension[34].

Latterly, In[35] revealed that there are two different category of entirely homogeneously rigid scrambled system shows weak mixing and proximality. Thus, we can say that.

Theorem 7. There are entirely homogeneously rigid scrambled system which is weak mixing and proximal[35].

Theorem 8. A topological dynamical system having positive entropy of ergodic invariant measure m. Then for this measure there exist a point b other than a of same space such that the ordered pair is asymptotic[36].

If the orbit of some fixed point is scrambled then the space containing this point is compact and the set containing these fixed points are scrambled as well the subset of the given space.

Theorem 9. Let f be a continuous function mapping between unit closed interval has positive topological entropy if it's iterated functional has an uncountable invariant scrambled set[37].

In[37] shown that.

Theorem 10. If the proper transitive dynamical system has a fixed point, then this system has dense Mycielski invariant scrambled subset K[37].

In 2010,[38] had done research work on invariant -scrambled sets and demonstrated that:

Theorem 11. If the properstrongly mixing dynamical system has a fixed point, then for any positive real number this system has dense Mycielski invariant -scrambled subset S[39].

The necessary and sufficient condition for transitive dynamical system has invariant -scrambled (subsets that it should not uniformly rigid.

Theorem 12. The necessary and sufficient condition for a propertransitive dynamical system has a dense Mycielski invariant -scrambled (, that this system has fixed point as well as this system not uniformly rigid[35].

Chaotic dynamical systems having Li-Yorke property facing strong limitations due to the absence of Cantor scrambled set in them. On the other hand,

Theorem 13. For any positive real number chaotic dynamical systems having Li-Yorke property have Cantor -scrambled set[31].

The notion of equicontinuity behaves totally different from sensitivity.

A topological dynamical system is equicontinuous if it preserves the distances. All such dynamical systems are simple in nature. Here, we have minimal systems exhibiting dichotomy.

Theorem 14. Dynamical systems minimal in nature possess either equicontinuity or sensitivity[23].

Any point which preserves the distance is termed as equicontinuous point. A transitive topological dynamical system having equicontinuous point is almost equicontinuous.

Here, we have transitive topological dynamical system exhibiting dichotomy.

Theorem 15. A transitive topological dynamical system is-

Almost equicontinuous (set of equicontinuous points correspondsto set of transitive points)

Sensitive[4,39].

In[39] had firstly informed that equicontinuous and uniform rigid property are interrelated in Ergodic theory this was the parallel proof of topological rigidity.

Theorem 16. Every topological dynamical system holds the property of uniformly rigidity if this system gave rise to an almost equicontinuous system[39].

Generally almost all equicontinuous topological dynamical system possess the property of uniformly rigidity.

Theorem 17. Every proper weak mixing topological dynamical system is Li-Yorke sensitive[2].

In[40] were the two mathematicians who had implemented the thoughts of probability theory for the development of a new definition of chaos termed as distributional chaos. An ordered doublet of the dynamical system is distributionally scrambled if

$\forall a > 0$, $\Phi*xy(a) = 1$, and

$\exists$ such that $\Phi xy(\ ) = 0$.

Collection of ordered doublet of distinct elements constitutes distributional scrambled set D, the subset of dynamical system. If the given set is uncountable subsequently the topological dynamical system is distributional chaotic. This infers that distributional chaotic topological dynamical system is stronger than the Li-Yorke chaotic topological dynamical system. Further, distributional -scrambled sets and distributional -chaos can be defined. It is observed that any distributional scrambled pair is scrambled, and therefore chaos. In[40] demonstrated-

Theorem 18. Every continuous function f having positive topological entropy, mapping between unit closed interval have distributional chaos[40]. In[41] had explored that every successive distributional chaotic dynamical systems are necessarily distinct.

Theorem 19. There are transitive topological dynamical system, "n ≥2 have successive distinct distributional chaos[41].

In 2013, Li and Oprocha demonstrated-

Theorem 20. There are topological dynamical systems which have the presence of n-distributional chaos and have the absence of (n + 1)-Li-Yorke chaos "n ≥2[42].

Recently, Dolezelova showed some results that are.

Theorem 21. There are topological dynamical systems containing countless external distributional scrambled collection having no scrambled triplet[43].

Theorem 22. There are unvarying Mycielski open set X in the complete transformation having extreme points which are disperse that constitute the set of scrambled doublet having no scrambled triplet[43].

In[7] had isolated the 3 important essential characteristics of system havening chaotic nature. For their study topological dynamics employed as important tool. Topological dynamical systems have chaos nature if the continuous function f defined over the systems possess given characteristic:

- f is topologically transitive, i.e., $\forall\ \phi \neq A, B\,(\text{open sets}) \subset X,\ \exists\ m \in \mathbb{N}\ \&\ f^m\,(A) \cap B \neq \phi$
- There are cases in which transitivity is an irreducibility condition.
- f is periodically dense. This state is also termed as an "element of regularity".
- f holds sensitivity based on preliminary situation.

When X = R, every topological dynamical system with sensitivity having chaos nature reckons on preliminary situation. These are the equivalent conditions for a dynamical system to be sensitive. Many natural systems such as earth's weather system having chaotic behavior can be studied through recurrence plots and Poincare maps the main platform of chaotic theory which can be applied in meteorology, physics, computer science etc. Due to the advancement of technology people in this world are connected via internet. Through which people can share their valuable information. But this information must be secured and this can be make possible using encryption and decryption method. The theory behind this method is chaos which plays an important role for this security[44]. There is another method known as cryptography, in which the safety of any process is based on the potency of the clue used. In this method the coordination is maintained by chaotic functions which are one more tool of chaos theory[45]. Also, data can be secured using fractal image generation method using chaos theory. Chaos is an impulsive behaviour that arises in the dynamical system[46].

The next section deals with the overview on Furstenberg families.

3. Furstenberg Family

In 1967[17], further extend the idea of families of subsets of non-negative integers (Z_+) initially proposed by two researchers Gottschalk and Hedlund[17] and gave new name to this as Furstenberg family denoted by F. Let C be the collection of all subsets of Z_+, then $C = C(Z_+) = \{F : F \subset Z_+\}$. A subset Fof given collection of non-negative integersis recall as Furstenberg family, iff the family F holds the property of hereditary upward. Mathematically, if $A \subset B \ \& \ A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$. Empty set and the whole dynamical space also represent this kind of family. In addition to this for the Furstenberg familyF, if f ÞF= C; and if Z_+ ÏF, then F= f. It had been observed that the family comprising of all subsets of non-negative integers of infinite cardinality also represent the familyF, referred by B[47]. Fis proper subset of C, if neither F¹f nor F¹C. Also, Fis said to be proper if the set of natural number is also the member of F, i.e., NÌF and fÏF. Several non-empty groups Cof subsets of non-negative integers create a family $\mathcal{F}(C) = \{A : A \subset \mathcal{Z}_+, A \in C\}$ [48].

The dual family of F which is also refers as Furstenberg family is given by $\mathcal{KF} = \{A : A \in C, \mathcal{Z}_+ - A \in \mathcal{F}\} = \{A : A \in C, A \cap A' \neq \phi \ \forall \ A' \in \mathcal{F}\}$. Obviously, dual of given collection of non-negative integers is non-empty also the dual of an empty set represents the same collectionC. The dual of any proper Furstenberg family is also proper. Also, $\mathcal{K}(\mathcal{KF}) = \mathcal{F}$ and $A, B \in \mathcal{F}$ (Furstenberg families) such that $A \subset B \Rightarrow \mathcal{KB} \subset \mathcal{KA}$. Also, dual family dual(B) of B refers a group comprises of members of Z_+ with finite elements. Furstenberg family [G] = {F ÌC: F Ì G, for some G ÎG} is the family generated by the subset G belonging to C. It refers minimal Furstenberg family containing G. It had been observed that, G= fÛ[G] = f, similarly fÎGÛ[G] = C. Therefore, [G] is proper iff G is non-empty as well as it does not contain empty set. If there is a denumerable member G of given collection C such that [G] = F, then the family generated by the given member is also denumerable. The dual of the Furstenberg family B is also countably generated[48]. Let consider F_1 and F_2 be any two Furstenberg families, then their product can be defined as $F_1 \cdot F_2 = \{F_1 \cap F_2 \colon F_1 \subset F_1,$

$F_2 \subset F_2\}$. A proper Furstenberg family F indicate complete, if κFÇFÌB. Thus, if κB∩FⵦF, then every F is said to be complete. Hence both the families B and κB are complete as well as Furstenberg family. Similarly, a Furstenberg family F is complete iff κF is complete. Also, if the family F is complete, then κBÌF. If F is proper family such that κF∩FⵦF, then F is known as filter[41].

Lemma 1. $\mathcal{F}$ is complete $\Rightarrow$ dual($\mathcal{B}$) $\subset \mathcal{F} \subset \mathcal{B}$. If the given family refers to a filter, then dual($\mathcal{B}$) $\subset \mathcal{F}$ ⵦ $\mathcal{F}$ is complete[3].

The family having infinite subsets of non-negative integers $\mathcal{Z}_+$ is denoted by $\mathcal{F}_{inf}$ and its dual $\kappa \mathcal{F}_{inf}$ denoted by $\mathcal{F}_{cf}$ is the aggregation of co-finite member of $\mathcal{Z}_+$. $\mathcal{F}$ refers to be complete having the attributes of proper set and holds the condition $\mathcal{F} \cap \ \ \mathcal{F} \subset \mathcal{F}_{inf}$ [14]. The finite sum S of any infinite sequence $\{x_n\}$ is referred as

$$S = \lim_{n \to \infty} \sum x_n = \left\{ \sum_{i \in \alpha} x_i : \phi \neq \alpha \subset \mathbb{N} \right\}.$$ If F $\subset \mathcal{Z}_+$, then F is termed as IP- set if $\exists \ \{x_n\} \in \mathcal{N}$ ⵦ FS$\{x_n\} \subset$ F and family of all IP-set is represented by $\mathcal{F}_{ip}$. Suppose F $\subset \mathcal{Z}_+$, then F is termed as thick set if ⵦn ⵦ $\mathcal{N}$ ⵦ some u_n ⵦ $\mathcal{Z}+$ such $\{u_n, u_n + 1, u_n + 2, \ldots, u_n + n\} \subset F$, i.e., it collection of randomly elongated positive integers. Again F $\subset \mathcal{Z}_+$ is syndetic set if $[m, m + \mathcal{N}] \cap F \neq \phi, m \in \mathcal{Z}_+$. Now, if $\mathcal{F}_t$ and $\mathcal{F}_s$ referring the families of randomly elongated positive integers and syndetic sets respectively, then $\mathcal{F}_s = \mathcal{F}_t$.

The next section deals with the review on Furstenberg family compatible with the dynamical System.

3.1 Furstenberg Family Compatible with Dynamical System

Let X be a complete space equipped with the distance function d and f be continuous functions defined over X holds the property of isomorphism, then the system (X, f) is a topological dynamical system (TDS)[13]. Let $\mathcal{F}$ be a Furstenberg family defined on the set of positive integers $\mathcal{Z}_+$ which the subset is of space X on which the metric d is defined. The distance between two non-empty subsets A, B the member of X is $d(A, B) = \inf \{d(a, b) : (a, b) \in A \times B\}$. If $d(A, B) > 0$, then A and Bare positively disjoint. For any dynamical system (X, f) the

entering time set of an element a into A is defined by $\mathcal{N}(a,A)=\left\{k\in\mathcal{Z}_+ : f^k(a)\in A, a\in X \ \& \ \phi\neq A\subset X\right\}$[39]. For G_1, G_2 $\subset$ X, $\mathcal{N}_f(G_1,G_2)=\left\{k\in\mathcal{Z}_+ : G_1\cap f^{-k}(G_2)\neq\phi\right\}=\left\{k\in\mathcal{Z}_+ : G_2\cap f^{-k}(G_1)\neq\phi\right\}$ referred as meeting time set or collection of common points of the sets G_1 and G_2. In particular, if $G_1 = \{a\}$, then $\mathcal{N}_f\left(\{a\},G_2\right)$ is merely expressed by $\mathcal{N}_f\left(a,G_2\right)$ which is recall as return time set from a to G_2. Suppose G_1 is the member of the space X then any element a of the space X is termed as F-binding element of the collection G_1, this is possible only if the return time set from a to the collection G_1 belongs to F. Here $\mathcal{F}\left(G_1,f\right)=\bigcup_{F\in\mathcal{F}}\bigcap_{k\in\mathcal{F}} f^{-k}\left(G_1\right)$ refer to F-binding group of the collection G_1.

Lemma 2. For any member G_1 of X and family $\mathcal{F}$, $\mathcal{F}\left(G_1,f\right)=\bigcup_{F\in\mathcal{F}}\bigcap_{k\in\mathcal{F}} f^{-k}\left(G_1\right)$ [48].

Any point is said to be an iterated point if its return time set contains innumerable elements. Let us now define the recurrence of Furstenberg family $\mathcal{F}$. Any element x of X is termed as $\mathcal{F}$-recurrent if for any neighbourhood G of x, N(x, G) $\in\mathcal{F}$. Thus, using the above result following lemma[2,22,49].

Lemma 3. Let X be TDS and x be any its member such that which satisfy the following conditions

- The given point will be minimal point iff it coincides with $\mathcal{F}$s-recurrent point;
- The given point will be iterated point iff it coincides with $\mathcal{F}_{ip}$-recurrent point[2,22].

Similarly, if G_1, $G_2 \subset$ X, of the dynamical system (X,f), the hitting time set of G_1 and G_2 is given by $\mathcal{N}\left(G_1,G_2\right)=\left\{k\in\mathcal{Z}_+ : f^k\left(G_1\right)\cap G_2\neq\phi, \phi\neq G_1, G_2\subset X\right\}$. If $G_1\subset X$, then $N(G_1,G_1)$ is termed as the return time set of U.

The next section deals with the short review on Transitive Points via Furstenberg Family.

3.2 Study of Transitivity via Furstenberg Family

Suppose the Furstenberg family $\mathcal{F}$ consists of set of positive integers $\mathcal{Z}_+$ contained in the space X such that $\mathcal{F}\subset\mathcal{B}$. Any member x of TDS (X,f) is termed as $\mathcal{F}$-transitive if $\left\{n\in\mathcal{Z}_+ : f^n(x)\in G_i, \forall \ \phi\neq G_i \ (\text{open})\subset X\right\}\in\mathcal{F}$. In this case the system referred as $\mathcal{F}$ - point transitive system. Study of such systems helps us in understanding the TDS. Transitive systems can be classified in different manner. Furstenberg had given the one of the renowned method using the concept of hitting time collection of two open members of the phase space X. This method reveals that the Furstenberg family $\mathcal{F}$ is, $\mathcal{F}$-transitive if $\mathcal{N}\left(G_1,G_2\right)\in\mathcal{F} \ \forall \ \phi\neq G_1, G_2 \ (\text{open})\subset X$. Also TDS is weakly mixing iff the collection of thick sets of the given space X is of transitive nature. Weak disjointness property is the second method employed for testing transitive character of a system. Two TDS $(X_1,f_1) \& (X_2,f_2)$ are weakly disjoint if $(X_1\times X_2, f_1\times f_2)$ holds transitivity[21]. Applying this property a unique result can be derived as (X,f) exhibit the characteristics of weak combination this is possible only in case the given space X behave differently from its weak mixing nature. Applying complexity of open covers of a system its transitive property can also be checked and this is the third method[50]. Let (X,f) be a TDS with exhaustible open-cover $\mathcal{O}$, suppose $N(\mathcal{O})$ is the small quantity of sub-cover of given open cover $\mathcal{O}$. The concept of embarking time collection of an element of a subset of the system is also used for showing transitive nature of a Furstenberg family. Thus for the family $\mathcal{F}$, (X,f) is $\mathcal{F}$-transitive, if $\mathcal{N}\left(x,G\right)\in\mathcal{F} \ \forall \ \phi\neq G \ (\text{open})\subset X$, i.e., (X,f) is transitive $\Leftrightarrow$ $\mathcal{N}\left(G_1,G_2\right)=\left\{k\in\mathcal{Z}_+ : f^k\left(G_1\right)\cap G_2\neq\phi, \phi\neq G_1, G_2\subset X\right\}\to\infty$. Thus, for any space X and Furstenberg family $\mathcal{F}$

- F holds transitivity if N(G1, G2) F
- F holds weak mixing property if $(X\times X, f\times f)$ is F-transitive.

Now using the above results following theorem holds[22,2].

Theorem 23. Every TDS system having weak mixing property is $\mathcal{F}_t$ –transitive[50].

For any TDS, X and Furstenberg family $\mathcal{F}$, the entering time set of any member of the given space X must belongs to the given family $\mathcal{F}$. This member is termed as $\mathcal{F}$-transitive point. Collection of these members of the

space X is represented as $\text{Trans}_{\mathcal{F}}(X,f)$ and if the collection is finite to some extent then the system is known as $\mathcal{F}$-point transitive. Similarly, the given TDS is termed as $\mathcal{F}$-point centre if every member of the space have entering time set which belongs to the given Furstenberg family $\mathcal{F}$. A following result can be derived from above concepts.

Lemma 4. Every TDS X with the Furstenberg family $\mathcal{F}$ exhibiting transitivity and having $\mathcal{F}$-point centre holds the condition $\text{Trans}_{\mathcal{F}}(X,f)=\text{Trans}(X,f)$ [50].

The next section deals with the short review on mixing property via Furstenberg family.

3.3 Study of Mixing Property via Furstenberg Family

All of us familiar with the weak mixing property of TDS (X,f) which infers that the system it holds the said property iff $(X\times X, f\times f)$ is transitive. Suppose (X,f) is a TDS with the Furstenberg family $\mathcal{F}$ be a defined over the set of non-negative integers having hereditary property. Now, for $\mathcal{F}=\mathcal{B}$, $\mathcal{B}$-transitivity (respectively. $\mathcal{B}$-mixing, $\mathcal{B}$-iteration) is the common transitivity (respectively weak combination, iteration).

Lemma 5. For any member x of the TDS (X,f)

- x is a minimal point in case the entering set of the given member is of syndetic nature.
- x is an iterated point in case the entering set of the given member have an IP-set.
- (X,f) is weak combination in case N(A,B) is thick set.
- (X,f) is strongly mixing in case N(A,B) is co-finite
- (X,f) is $\mathcal{F}$-mixing in case the Furstenberg family $\mathcal{F}$ holds transitivity and of weakly mixing nature [41].

Suppose (X,f) is TDS with some distance function defined over it. For any set of non-negative integers G contained in some open ball $\mathcal{B}$, the ordered duple (a,b) belonging to the space X is G-proximal in case $\liminf_{\substack{k\in G \\ n\to\infty}} d\left(f^{k}(a), f^{k}(b)\right)=0$. Here, $P_{G}=\left\{(a,b):(a,b)\in X\times X\right\}$ is a G-proximal relation and $P_{G}(x)$ the G –proximal cell at x.

Lemma 6. Suppose (X,f) is a TDS, where X is entirely steady T_{2}-space; so.

- System will have weak amalgamating property in case the return time set is thick and generates a filter.
- TDS (X,f) for any Furstenberg family $\mathcal{F}$ will have $\mathcal{F}$-mixing in case the given system holds the property of weak mixing and $\mathcal{F}$-transitivity [41].

Also the family $\mathcal{F}$ creates filter in case its dual have the Ramsey attribute, means that i.e. if $\bigcup G_{i}\in\overline{\mathcal{F}}(\text{dual of } \mathcal{F})\Rightarrow G_{i}\in\overline{\mathcal{F}}$ for atleast one value of i. From the given concepts following result can be explored.

Theorem 24. For any $\text{TDS}(X,f)$ and complete Furstenberg family $\mathcal{F}$, the system having $\mathcal{F}$-mixing indicates that $\forall A\in\mathcal{F}$ the proximal cell of every element of the space have neighbourhood will be contained in the space X [41].

Theorem 25. For any minimal $\text{TDS}(X,f)$ and complete Furstenberg family $\mathcal{F}$, the given system holds $\mathcal{F}$-mixing indicates that $\forall A\in\mathcal{F}$ the proximal cell of every element of the space have neighbourhood will be contained in the space X [41].

Using given result the following corollary can be checked.

Corollary 1. For every set belonging to the open ball $\mathcal{B}$ and the proximal cell of every element of the space have neighbourhood contained in the space X, then the minimal $\text{TDS}(X,f)$ is strongly mixing [41].

The non-trivial Furstenberg family exhibiting $\mathcal{F}$-transitivity and $\mathcal{F}$-mixing is same as transitive and weakly mixing. Thus, we can conclude that the given axioms holds [22,2].

The next section deals with the short review on Sensitivity via Furstenberg Family.

3.4 Study of Sensitive Nature of Furstenberg Family

A $\text{TDS}(X,f)$ have sensitivity if for any positive integers δ there exist two distinct points a and b in the open subsets of X such that $d\left(f^{k}(a), f^{k}(b)\right)>\delta$ for some $\kappa\in\mathbb{N}_{0}$. In other words there would be one moment arises

when $\exists a \neq b \in \mathcal{B} \subset X$ whose trajectories are apart from a reference point[51].

Proposition 1. For any TDS (X, f).

- The given system has sensitivity.
- For $\delta > 0 \exists a \neq b \in \mathcal{B} \subset X$ such that $\overline{\lim_{n \to \infty}} d\left(f^k(a), f^k(b)\right) > \delta$ for some $\kappa \in \mathbb{N}_0$.
- For $\varepsilon > 0 \exists a \neq b \in G$ (open) $\subset X$ such that $d\left(f^k(a), f^k(b)\right) > \varepsilon$ for some $n \in \mathbb{Z}_+$.
- For $\delta > 0 \exists a \neq b \in G$ (open) $\subset X$ such that $\overline{\lim_{n \to \infty}} d\left(f^n(a), f^n(b)\right) > \delta$ [51].

Thus, a TDS (X, f) have sensitivity iff $\exists \varepsilon > 0$ and in any open subset G of X have two different points whose curves are much separated with the minimum distance ε. For any TDS (X, f) and Furstenberg family $\mathcal{F}$ the system exhibits $\mathcal{F}$-sensitivity in case for any $\varepsilon > 0$ there exist two distinct points a and b in the open subsets of X such that $\left\{k \in \mathcal{Z}_+ : d\left(f^k(a), f^k(b)\right) < \delta\right\} \in \mathcal{F}$ for some $\kappa \in \mathbb{N}_0$. Such that for every $x \in X$ and every open neighbourhood N of x there exists $y \in N$ such that $\left\{n \in \mathbb{Z}_+ : d\left(f^n(x), f^n(y)\right) < \delta\right\} \in \mathcal{F}$.

Also for two different Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$ the topological dynamical system (X, f) is ($\mathcal{F}_1$, $\mathcal{F}_2$)-sensitive if $\exists \delta > 0$ such that for every $a \in X$ is a limit of points for $b \in X \left\{k \in \mathbb{Z}_+ : d\left(f^k(a), f^k(b)\right) < \delta\right\} \in \mathcal{F}_1$ while for $\varepsilon > 0 \left\{k \in \mathbb{Z}_+ : d\left(f^k(a), f^k(b)\right) > \varepsilon\right\} \in \mathcal{F}_2$.

Also, TDS is weakly $\mathcal{F}$-sensitive if for any $\mathcal{F}$-sensitive constant $\varepsilon \in \mathcal{Z}_+$ and the distinct pair (a, b) is not $\mathcal{F}$-ε-asymptotic where a, b $\in N$ (open neighbourhood) $\subset X$. Thus, TDS is weakly $\mathcal{F}$-sensitive if for any $\mathcal{F}$-sensitive constant $\varepsilon \in \mathcal{Z}_+$ and the distinct pair (a, b) is not $\mathcal{F}$-ε-asymptotic where a, b $\in N$ (open neighbourhood) $\subset X$. Here are some results based on the learnt concept.

Theorem 26. For Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$ such the dual of one family is contained in the other, then the system sensitivity of the system w.r.t. one family implies its sensitivity w.r.t. other family also[51].

Corollary 2. Every TDS is weakly $\mathcal{F}$-sensitive iff it is $\mathcal{F}$-sensitive for any filter-dual $\mathcal{F}$[51].

Theorem 27. Every TDS is weakly $\mathcal{F}$-sensitive for any filter-dual $\mathcal{F}$ compatible with $(X \times X, f \times f)$ [51].

A TDS is said to be accessible if $d\left(f^k(a), f^k(b)\right) < \varepsilon$ for every $\varepsilon > 0$, $a \in G_1(\text{open}) \subset X, b \in G_2(\text{open}) \subset X$ and $k \in \mathbb{N}$.

The next section deals with the short review on Proximities and Distality via Furstenberg Family.

3.5 Study of Proximities and Distality via Furstenberg Family

Study of asymptotic attributes of an ordered tuples is an important tool for exploring TDS. The ordered tuples (a, b) the member of the space X is said to be proximal if $\underline{\lim_{n \to \infty}} d\left(f^n(a), f^n(b)\right) = 0$. Collection of these tuples is represented by $P(X, f)$. The given pair of points behaves asymptotically if $\lim_{k \to \infty} d\left(f^k(a), f^k(b)\right) = 0$.

Collection of these tuples is represented by $Asym(X, f)$. If elements of the ordered pair are distinct then the tuples refer to a non-trivial pair. The set of proximal tuples obey reflexive, symmetric, f-invariant relation while disobey transitive relation. The ordered tuples are distal if they are not proximal. Any pair of points is said to a Li–Yorke if it obey proximality while disobey asymptotic behaviour. Any point of the system referred as returning point if there is an increasing sequence converges to the same point. Any ordered tuples of returning point is of strong Li–Yorke nature which infers Li–Yorke pair. A system whose ordered tuples have the lack of proximity pairs indicate a distal pairs that may be almost or semi-distal respectively. Also almost distal system infers semi-distal[52].

Suppose a and b be any two distinct points of the TDS (X, f).

- Suppose $B \subset \mathbb{Z}_+ \Rightarrow$ ordered tuples (a, b), said to be B-proximal if $\liminf_{n \to \infty} Bd\left(f^n(a), f^n(b)\right) = 0$ Also, (a, b) is said to be B-asymptotic in case $\lim_{n \to \infty} Bd\left(f^n(a), f^n(b)\right) = 0$. The same ordered tuples (a, b) referred as B-distal in case it is not B-proximal. Collection of all ordered tuples of B-proximal, B-asymptotic, A-distal are mainly denoted

by $PB(X,f), AB(X,f)$ and $DB(X,f)$ respectively.

- The ordered tuples (a,b) referred as $\mathcal{F}$-proximal in case

$$N((a,b),\Delta\varepsilon)\subset\mathcal{F} \; \forall\varepsilon>0 \text{ where } (a,b)\subset B\subset N$$

and $\Delta\varepsilon=\{(a,b)\in X: d(a,b)<\varepsilon\}$.

The collection of all $\mathcal{F}$-proximal tuples are being represented as $N((a,b),\Delta\varepsilon)\subset P_{\mathcal{F}}(X,f)$ or $P_{\mathcal{F}}$

The following result can be drawn from the above concepts:

Corollary 3.

- The ordered tuples (a,b) referred as $\mathcal{F}$-proximal under the condition that $\mathcal{F}$ is contained in its dual, $\overline{(f\times f)}^F(a,b)\cap\Delta_x\neq\phi$ in case the ordered tuples belongs to the proximal pairs. Also

$$P_{\mathcal{F}}(X,f)=P_{\mathcal{F}}=\bigcap_{\varepsilon>0}\bigcup_{F\in\mathcal{F}}\bigcap_{n\in F}(f\times f)^{-F}\Delta_{\varepsilon}$$

- For every Furstenberg family $\mathcal{F}$ which is a filter its every ordered pair is k$\mathcal{F}$-proximal such that for any point p $\in$H($\mathcal{F}$), pa = pb. The same ordered pair is $\mathcal{F}$-proximal in case for every point p $\in$H($\mathcal{F}$), pa = pb. For $\mathcal{F}$= k$\mathcal{B}$, the ordered pair (a,b) is proximal in case the ordered pair is $\mathcal{B}$-proximal for every point p $\in$H (k$\mathcal{B}$) such that pa = pb. For every point p $\in$H (k$\mathcal{B}$), where pa = pb the ordered pair (a,b) shows asymptotic behaviour if it is k$\mathcal{B}$proximal[54].

Two different Furstenberg family each defining on the set of positive integers showing upper hereditary property constitute a couple known as Furstenberg Family Couple[54], which helps us in exploring the concept of chaos in Furstenberg family.

Next we have discussion on chaotic nature of dynamical system will be explored with respect to Furstenberg family.

4. Chaos in Furstenberg Families

TDS and Furstenberg families are much interrelated with each other. In2009, Xiong, Lu and Tan[54] had described the chaotic theory via Furstenberg families. Further chaos

such as Li-Yorke and various category of distributional-chaos becomes a tool for the study of chaotic nature of Furstenberg families' sense.

G $\subset$X referred as $\mathcal{F}$-scrambled set of TDS (X,f) in case the ordered pair of distinct elements that are the members of G constituting $\mathcal{F}$-scrambled pair. For any positive real number δ >0, the same set G $\subset$X with same attributes is referred as strongly $\mathcal{F}$-scrambled provide $\mathcal{F}$-δ-scrambled pair.

(X,f) referred as $\mathcal{F}$-chaotic, if collection of all $\mathcal{F}$-scrambled pairs provide G_δ set that are dense in the Cartesian space composed of X. For any δ >0, (X,f) have strong $\mathcal{F}$-chaotic, if collection of all $\mathcal{F}$-δ-scrambled pairs are G_δ set which are again dense in the Cartesian space composed of X.

Result: (X,f) is commonly $\mathcal{F}$-chaotic (or, strongly $\mathcal{F}$-chaotic) where X possess completeness and separablity having no isolated points, which infers $\mathcal{F}$-scrambled collection X (or strongly $\mathcal{F}$-scrambled set, respectively) is a Mycielski set[54,55].

Suppose be a dynamical system. For any two Furstenberg families $\mathcal{F}_1, \mathcal{F}_2$ and (X,f) the ordered doublet (a,b) of the space X is ($\mathcal{F}_1, \mathcal{F}_2$)-scrambled if

- $\left\{n\in\mathcal{Z}_+ : d(f^n(a),f^n(b))<\varepsilon\right\}\in\mathcal{F}_1$ ”ε>0 and

- \$ some d> 0 such that $\left\{n\in\mathcal{Z}_+ : d(f^n(a),f^n(b))<\delta\right\}\in\mathcal{F}_2$.

Any subset S of the space X is ($\mathcal{F}_1, \mathcal{F}_2$)-scrambled if it have distinct elements that create ($\mathcal{F}_1, \mathcal{F}_2$)-scrambled pair. In case S is uncountable, (X,f) will be ($\mathcal{F}_1, \mathcal{F}_2$)-chaotic. Also, for uniformly detached unvaryingδ, all non-diagonal pairs of the subset S, form ($\mathcal{F}_1, \mathcal{F}_2$)-$\delta$-scrambled sets and ($\mathcal{F}_1, \mathcal{F}_2$)-$\delta$-chaos. Let F $\subset\mathbb{Z}_+$, then $\Delta(F)$ (upper density of F) expressed as

$$\Delta(F)=\limsup_{n\to\infty}\frac{1}{n}\left(F\cap\{0,1,2,...,n-1\}\right).$$

Furstenberg family $\mathcal{F}(a) \; \forall a\in[0,1]$ is given by

$$\overline{\mathcal{F}}(a)=\left\{F\subset\mathcal{Z}_+ : F\to\infty \text{ and } \Delta(F)\geq a\right\}.$$ The chaotic behaviour of Li-Yorke and distributional chaos via Furstenberg families is characterized using the axiom.

Proposition 2. For any ordered doublet (a,b) of (X,f)

- (a,b)-scrambled iff it is $(\overline{\mathcal{F}}(0),\overline{\mathcal{F}}(0))$-scrambled.
- (a,b)-distributionally scrambled type 1 iff it is $(\overline{\mathcal{F}}(1),\overline{\mathcal{F}}(1))$-scrambled.
- (a,b)-distributionally scrambled type 2 iff it is $(\overline{\mathcal{F}}(1),\overline{\mathcal{F}}(\varepsilon))$-scrambled

for positiveε[48].

$\mathcal{F}$ is well-matched with (X,f) if $\{x\in X:\{n\in\mathcal{Z}_+:f^n x\in O\}\in\mathcal{F}\}=G_\delta\subset X\ \forall\phi\neq O(\text{open})\subset X$.

The condition for given concept can be shown as:

Theorem 28. In (X,f), for any fixed point $a\in X$ such that $A\cap\{a\}=\phi$ and $\phi\neq A(\text{closed invariant})\subset X$ and $\bigcup_{i=1}^{\infty}f^{-i}(a)$ is dense in X. Then for any $\mathcal{F}_1$and$\mathcal{F}_2$ compatible with the system $(X\times X,f\times f)$ $\exists$dense Mycielski $(\mathcal{F}_1,\mathcal{F}_2)$-δ-scrambled set $\forall$δ> 0.

Generally (X,f) is $(\overline{\mathcal{F}}(1),\overline{\mathcal{F}}(1))$-δ-chaotic $\forall$δ> 0[42, 47].

Any ball$\mathcal{B}$ in space X having distinct elements constituting Banach scrambled pair, is said to be Banach scrambled set that holds proximality. Also, every proximal tuples is Banach proximal for functions defined on a space having null topological entropy[57].

Theorem 29. Any continuous functions defined over unit closed interval having Li-Yorke chaos have Cantor Banach scrambled set[57].

Theorem 30. Any continuous functions defined over unit closed interval have positive topological entropy iff $\exists$ Banach scrambled set C(Cantor set) subset of unit closed intervaland the iterated value of the function over the given interval contained in C[57].

(X,f) in which X is a scrambled set then in such system every pair is syndetically proximal[38]. In addition to X may be Banach scrambled set[57].

Theorem 31. (X,f) is a transitive with X as Banach scrambled set[57].

5. Everywhere Chaos and Equicontinuity via Furstenberg Families

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Furstenberg families. (X,f) is $(\mathcal{F}_1,\mathcal{F}_2)$–everywhere chaotic in case in Furstenberg Family couple $(\mathcal{F}_1,\mathcal{F}_2)$, $\mathcal{F}_1$ is sensitive and $\mathcal{F}_2$ is accessible. For the better understanding of the concept of $(\mathcal{F}_1,\mathcal{F}_2)$-everywhere chaotic let us understand the concept of hyperspace. The hyperspace of a topological dynamical space X, is defined by 2^X = {C: $\Phi\neq$C $\subset$X and C is compact} through the Hausdorff distance function d_H, i.e.,

$$d_H(G_1,G_2)=\inf\{\varepsilon>0:[G_1]_\varepsilon\supset G_2,[G_2]_\varepsilon\supset G_1\},$$

where G_1, $G_2\in 2^X$.

In[58] had given the evidence of the following result.

Theorem 32. Suppose (X,f) have no inaccessible point. Then (X,f) is everywhere chaotic

if for δ>0 and $\forall\varepsilon$ >0, $\exists$G $\in 2^X$ and Cantor set C $\subset$X having distinct elements a, b such that $d_H(G,C)<\varepsilon$ implies $\liminf\limits_{n\to\infty}d(f^n(a),f^n(b))=0$ and $\limsup\limits_{n\to\infty}d(f^n(a),f^n(b))>\delta$ [58].

Main consequence of everywhere chaos is depends on given result.

Theorem 33. Suppose space X of the system (X,f) have no inaccessible point.

- (X,f) is weakly $(\mathcal{F}_1,\mathcal{F}_2)$–everywhere chaotic.
- (X,f) is $(\mathcal{F}_1,\mathcal{F}_2)$–everywhere chaotic.
- There is δ>0 such that for each ε >0 and each G $\in 2^X$ $\exists$ Cantor set C in X having distinct points where d_H(G,C) <ε, then
- for any ε> 0, {n $\in\mathbb{Z}_+$: d(f^nx, f^ny) <ε} $\in\mathcal{F}_1$, and
- $\exists$ some δ> 0 such that {n $\in\mathbb{Z}_+$: d(f^nx, f^ny) >δ} $\in\mathcal{F}_2$ [58].

Then (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). If $\mathcal{F}_1$ and $\mathcal{F}_2$are compatible with (X × X, f × f), and $\mathcal{F}_1$ is a filter dual, then all assertions are equivalent. The correlation between k$\mathcal{F}$-equicontinuity[3] and weakly $\mathcal{F}$-sensitivity is the main point of attraction in transitive dynamics.

Any member of the space is an $\mathcal{F}$-equicontinuity member[3] in case $\forall \varepsilon > 0$ $\exists \delta > 0$ and $d(a,b) < \delta$ this implies $\left\{ k \in \mathbb{Z}_+ : d\left(f^k(a), f^k(b)\right) < \varepsilon \right\} \in \mathcal{F}$, where $\mathcal{F}$ is a Furstenberg family.

Remark: In case a TDS shows k$\mathcal{B}$-equicontinuous, then this TDS behave usually equicontinuous. Similarly, if any member of the space shows k$\mathcal{B}$-equicontinuity then this member exibhit equicontinuity.

Proposition 3. For any compact metric space in topological dynamical system having $\mathcal{F}_1$ and $\mathcal{F}_2$ as Furstenberg families and $\mathcal{F}_1$ $\mathcal{F}_2 \subset \mathcal{F}_2$. Thus, all the member of the system having $\mathcal{F}_1$-equicontinuity, then that system is $\mathcal{F}_2$-equicontinuous[23].

Corollary 4. For any compact metric space in a TDS, having $\mathcal{F}$ as filter. Then TDS is $\mathcal{F}$-equicontinuous iff every member of X shows $\mathcal{F}$-equicontinuity[23].

Auslander-Yorke[23] had provided the following result:

Proposition 4. For any compact metric space $(X\ d)$ in a transitive TDS (X, f), suppose kF is an invariant Furstenberg family having translation. Thus,

TDS having no k$\mathcal{F}$-equicontinuity point infers that the system is weakly $\mathcal{F}$-sensitive.

TDS having k$\mathcal{F}$-equicontinuity point infers that collection of all k$\mathcal{F}$-equicontinuous points of TDS is a dense G_δ set[23].

Remark: If TDS have k$\mathcal{F}$-equicontinuity, then it is not $\mathcal{F}$-sensitive.

Proposition 5. For any compact metric space in a TDS (X, f), suppose $\mathcal{F}$ is a filter, and $\mathcal{F} \subset \mathcal{B}$. For any $\mathcal{U} \in \mathcal{C}_X^0$ $\exists$ F $\in \mathcal{F}$ such that $\mathcal{C}_{\mathcal{F}}(\mathcal{U}) < +\infty$ then (X, f) is $\mathcal{F}$-equicontinuous[23].

6. References

1. Adler RL, Konheim AG, McAndrew MH. Topological entropy. Trans Amer Math Soc. 1965; 114:309–19.
2. Akin E, Kolyada S. Li-Yorke sensitivity. Nonlinearity. 2003; 16(4):1421–33.
3. Akin E. Recurrence in topological dynamics: Furstenberg families and Ellis actions. The University Series in Mathematics. New York: Plenum Press; 1997.
4. Akin E, Auslander J, Kenneth B. When is a transitive map chaotic? Conference on Convergence in Ergodic Theory and Probability; Columbus, OH. 1996; p. 25–40.
5. Ruelle D. Mathematical problems in theoretical physics. Proceedings of International Law Conference Univ. Rome, Rome, of Lecture Notes in Physics; Berlin, Germany. 1978. p. 341–60.
6. John G. Sensitive dependence to initial conditions for one-dimensional maps. Comm Math Phys. 1979; 70(2):133–60.
7. Devaney RL. An Introduction to Chaotic Dynamical Systems. 2nd ed. Redwood City, CA: Addison-Wesley; 1989.
8. Anzelm I. Independence and scrambled sets for chaotic mappings. The Mathematical Heritage of C. F. Gauss. River Edge, NJ: World Sci. Publ; 1991. p. 372–8.
9. Jincheng X, Zhongguo Y. Chaos caused by a topologically mixing map. Dynamical Systems And Related Topics Adv. River Edge, NJ: World Sci. Publ.; 1991. p. 9550–72.
10. Bruno S. Stability properties of flows with pure point spectrum. J London Math Soc. 1982; 26(2):451–64.
11. Kathleen T, Alligood A, Tim D, Sauer S, Yorke JA. One dimensional map. Two Dimensional Maps in Chaos: An Introduction to Dynamical Systems. New York: Springer; 2000; p. 6–52.
12. Brin M, Stuck G. Examples and basic concepts. Introduction to Dynamical Systems. 2nd ed. USA: Cambridge University Press; 2003; 1(1):1–17.
13. Kriheli M. Chaos Theory and Dynamical System. New York. 2011; 4(3):366–93.
14. Li J, Ye X. Recent development of chaos theory in topological dynamics. Acta Math Sin, Eng Ser. 2015; 32(1):83–114.
15. Wen H, Song SY, Xiangdong X. Mixing and proximal cells along sequences. Nonlinearity. 2004; 17(4):1245–60.
16. Sharma P, Nagar A. Topological dynamics on hyperspaces. Applied General Topology. 2010; 11(1):1–19.
17. Helbig GW, Hedlund GA. Topological dynamics, American mathematical society colloquium publications. American Mathematical Society, Providence, R. I. 1955; 36(1):890–5.
18. Auslander J. On the proximal relation in topological dynamics. Proc Am Math Soc. 1960; 11(6):890–5.
19. Auslander J. Minimal Flows and their Extensions. North-Holland Mathematics Studies. 1988; 153:23.
20. Auslander J. Minimal flows with a closed proximal cell. Ergodic Theory Dynamical Systems. 2001; 21(3):641–5.
21. Peleg R. Weak disjointness of transformation groups. Proc Amer Math Soc. 1972; 33:165–70.
22. Furstenberg H. Recurrence in Ergodic Theory and Combinatorial Number Theory. M. B. Porter Lectures. Princeton, NJ: Princeton University Press; 1981.
23. Auslander J, Yorke Y. Interval maps, factors of maps and chaos. Tohoku Math J. 1980; 32(2):177–88.
24. Barreira L, Valls C. Dynamical Systems an Introduction. 2nd ed. London: Springer; 2012. p. 27–54.
25. Li TY, Yorke JA. Period three implies chaos. Amer Math Monthly. 1975; 82(10):985–92.

26. Blanchard F, Pairs OLY. Journal fur dies Reine und Angewandte Mathematik. 2002; 547:51–68.

27. Smital J. A chaotic function with some extremal properties. Proc Amer Math Soc. 1983; 87(1):54–6.

28. Jankova K, Smital J. A characterization of chaos. Bull. Austral Math Soc. 1986; 34(2):283–92.

29. Xiong J. A chaotic map with topological entropy. Acta Math Sci. 1986; 6(4):439–43.

30. Smital J. Chaotic functions with zero topological entropy. Trans Amer Math Soc. 1986; 297(1):269–82.

31. Blanchard F, Huang W, Snoha L. Topological size of scrambled sets. Colloq Math. 2008; 110(2):293–361.

32. Huang W, Lu P, Ye X. Measure-theoretical sensitivity and equicontinuity. Israel J Math. 2011; 183(1):233–83.

33. Mai J. Continuous maps with the whole space being a scrambled set. Chinese Sci Bull. 1997; 42(19):1603–6.

34. Huang W, Ye X. Homeomorphisms with the whole compact a being scrambled sets. Ergodic Theory Dynam Systems. 2001; 21(1):77–91.

35. Fory M. Invariant scrambled sets, uniform rigidity and weak mixing. Israel J Math. 2016; 211(1):447–72.

36. Blanchard F, Host B, Ruette S. Asymptotic pairs in positive-entropy systems. Ergodic Theory Dynam Systems. 2002; 22(3):671–86.

37. Yuan D, Lu J. Invariant scrambled sets in transitive systems. Adv Math. 2009; 38(3):302–8.

38. Balibrea F, Guirao JLG, Oprocha P. On invariant d-scrambled sets, Internat. J Bifur Chaos Appl Sci Engrg. 2010; 20(9):2925–35.

39. Eli G, Benjamin W. Sensitive dependence on initial conditions. Nonlinearity. 1993; 6(6):1067–75.

40. Schweizer B, Smital J. Measures of chaos and a spectral decomposition of dynamical systems on the interval. Trans Amer Math Soc. 1994; 344(2):737–54.

41. Tan F, Fu H. On distributional n-chaos. Acta Math Sci Ser B. 2014; 34(5):1473–80.

42. Jian L, Piotr O. On n-scrambled tuples and distributional chaos in a sequence. J Difference Equ Appl. 2013; 19(6):927–41.

43. Jana D. Scrambled and distributionally scrambled n-tuples. J Difference Equ Appl. 2014; 8(1):1169–77.

44. Gulshan K, Praveen P, Rahul S, Mritunjay Kumar R. Chaotic image encryption technique based on IDEA and discrete wavelet transformation. Indian Journal of Science and Technology. 2016 Apr; 9(15):1–6.

45. Gautam A, Gaba GS, Miglani R, Pasricha R. Application of chaotic functions for construction of strong substitution boxes. Indian Journal of Science and Technology. 2015 Oct; 8(28):1–5.

46. Thamizhchelvy K, Geetha G. Data hiding technique with fractal image generation method using chaos theory and watermarking. Indian Journal of Science and Technology. 2014 Sep; 7(9):1271–78.

47. Cheng XJ. Furstenberg family and chaos. Science in China Press. 2007; 50(9):1325–33.

48. Li J, Tsun WW. Transitive points via Furstenberg family. Topology and its Applications. 2011; 158:2221–31.

49. Eli G, Xiangdong Y. Local entropy theory. Ergodic Theory Dynam Systems. 2009; 29(2):321–56.

50. Furstenberg H. Disjointness in Ergodic theory, minimal sets and a problem in Diophantine approximation. Math Systems Theory. 1967;1–49.

51. Huoyun W, Jincheng J, Feng T, Furstenberg F. Families and Sensitivity. 2009; 424:194–7.

52. Song S. Proximity and distality via Furstenberg families. Topology and its Applications. 2006; 153(12):2055–72.

53. Wen H, Song SY, Xiangdong X. Mixing and proximal cells along sequences. Nonlinearity. 2004; 17(4):1245–60.

54. Mycielski J. Independent sets in topological algebras. Fund Math. 1964; 55(1):139–47.

55. Keynes HB, Robertson JB. Eigen value theorems in topological transformation groups. Trans Am Math Soc. 1969; 139:359–69.

56. Li J. Chaos and entropy for interval maps. J Dynam Differential Equations. 2011; 23(2):333–52.

57. Li J, Tu S. On proximality with Banach density one. J Math Anal Appl. 2014; 416(1):36–51.

58. Kato H. Everywhere chaotic homeomorphisms on mainifolds and k-dimensional Menger manifolds. Topology and its Applications. 1996; 72(1):1–17.