Improved Extended Runge-Kutta-like Method for Solving First Order IVPs

Yusuf Dauda Jikantoro*, Fudziah Ismail, Norazak Senu and Mohamed Suleiman

Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400
Serdang Selangor, Malaysia; jdauday@yahoo.ca, fudziah_i@yahoo.com.my, norazak@upm.edu.my

Abstract

In this research, we proposed a family of improved extended Runge-Kutta-like methods which incorporate the function as well as the derivative of the function for the numerical integration of autonomous and non-autonomous ordinary differential equations. The proposed methods are more accurate than the existing methods in the literature and acquire bigger regions of stability. Numerical examples illustrating the computational accuracy are presented and the stability regions are also shown.

Keywords: Absolute Error, Absolute Stability, Extended Runge-Kutta, Ordinary Differential Equations

1. Introduction

In nature, most of the real life situations are modeled as differential equations. The importance of the solutions to these equations cannot be over emphasized. The best methods that provide exact solutions to the equations are the analytical methods. Unfortunately, only a few of these equations can be solved by analytical methods, hence there is a need to approximate the solutions using numerical methods.

One of the best numerical methods for approximating the solutions of differential equations is Runge-Kutta method. This paper is concerned with the following system of differential equations

$$y'(x) = f(x, y(x)) \qquad (1)$$

A lot of research has been done on trying to improve the efficiency of Runge-Kutta method. One of the popular way of improving the order of accuracy of Runge-Kutta method is by increasing the number of terms in the Taylor series expansion. This in turn increases the number of function evaluation accordingly, see Butcher[1] and Dormand[2]. As a result, researchers devised various ways of improving the order of Runge-Kutta method with reduced functions evaluation. In line with this, Goeken

and Johnson[3] proposed a class of Runge- Kutta method with higher derivatives approximations for the third and fourth-order method. Xinyuan[4] presented a class of Runge-Kutta formulae of order three and four with reduced evaluation of functions for autonomous first order differential equation. Phohomsiri and Udwadia[5] constructed an accelerated Runge- Kutta integration schemes for the third-order method using two functions evaluation per step for integrating autonomous Ordinary Differential Equations (ODEs). In their work, Xinyuan and Jianlin[6] proposed extended Runge-Kutta-like formulae for integrating autonomous system of ODEs. Udwadia and Farahani[7] developed the accelerated Runge-Kutta methods for higher orders. The set back of the various methods mentioned above is that they are capable of integrating autonomous system of ODEs only. As an improvement to this, Rabieiand Ismail[8] developed a third order Improved Runge-Kutta method for solving ordinary differential equations with two and three stages. This motivated us to propose a family of extended Runge-Kutta-like methods, which is directly based on the methods developed in[6] of the form

$$y_{n+1} = y_{n+} + \sum_{i=1}^{m} (hb_i k_{i1} + h^2 c_i k_{i2}) \qquad (2)$$

where

$$k_{i1} = f\left(x_n + \overline{c_i}h, \qquad y_n + h\sum_{s=1}^{i-1}a_{is}k_{s1}\right)$$ (3)

$$k_{i2} = f'\left(x_n + \overline{c_i}h, \qquad y_n + h\sum_{s=1}^{i-1}a_{is}k_{s2}\right)$$

And $c_i \neq 0$. Obviously, k_{i2} is the derivative of k_{i1}.

The paper is organized as follows: The general derivation of the methods is presented in section two, followed by the derivation of the methods in section three. The stability analysis of the methods is given in section four. Numerical examples are given in section five to illustrate the claimed accuracy and stability of the new methods and finally the last section gives the conclusion of the research.

2. Construction of the Methods

The derivation of the propose method is not farfetched from the derivation of classical Runge-Kutta method. The Taylor series method is given by

$$y(x_{n+1}) = y(x_n) + h\Delta(x_n, y(x_n), h)$$ (4)

Definition: Local truncation error of a numerical method is defined as the amount by which true solution fails to satisfy the numerical method, see[2].

Suppose we re-write eqn. (2) as

$$y_n + h\varphi(x_n, y_n, h) - y_n = 0$$

$$\varphi(x_n, y_n, h) = \sum_{i=1}^{m}(b_i k_{i1} + hc_i k_{i2})$$

where,

Then, by the definition above

$$t_{n+1} = y(x_n) + h\varphi(x_n, y(x_n), h) - y(x_{n+1}).$$ (5)

where, t_{n+1} is the local truncation error of the method in (2) Substituting eqn. (4) into (5) gives

$$t_{n+1} = y(x_n) + h\varphi(x_n, y(x_n), h) - \{y(x_n) + h\Delta(x_n, y(x_n), h)\},$$

$$t_{n+1} = y(x_n) + h\varphi(x_n, y(x_n), h) - \{y(x_n) + h\Delta(x_n, y(x_n), h)\},$$

$$t_{n+1} = h\{\varphi(x_n, y(x_n), h) - \Delta(x_n, y(x_n), h)\},$$ (6)

The task now is to obtain the complete expression for eqn. (6), which is done by obtaining the Taylor series expansions of Δ and φ about h in their scalar forms. When this is accomplished and the expansions are truncated at $p+1$ terms then,

$$t_{n+1} = h\{\varphi(x_n, y(x_n), h) - \Delta(x_n, y(x_n), h)\} + O(h^{p+1}),$$ (7)

where $f' = f_x + f_y f$.

2.1 Order Condition

The choice of the order of Runge-Kutta method is best done by minimizing its local truncation error. Therefore, if we take

$$h\{\varphi(x_n, y(x_n), h) - \Delta(x_n, y(x_n), h)\} = 0,$$ (8)

then t_{n+1} reduces to $O(h^{p+1})$

The equation in (8) is *pth* order condition for the propose method and it is a set of nonlinear equations with $\{b_i, c_i, \overline{c_i}, a_{is}\}$ set of unknown parameters. The next task is to solve the set of the nonlinear equations to obtain the values of the unknowns that satisfy eqn. (8). This can easily be achieved with the aid of maple package. The general *pth* order method is shown in a Butcher table (Table 1) where the stage of the method is *m*:

Table 1. Coefficients of the *pth* order method

$\overline{c_i}$	a_{ij}				
0	**0**				
$\overline{c_2}$	a_{21}	0			
$\overline{c_3}$	a_{31}	a_{32}	0		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	0	
$\overline{c_m}$	a_{m1}	a_{m2}		a_{mm-1}	0
b_i	b_1	b_2	b_3	$\cdots$	b_m
c_i	c_1	c_2	c_3	$\cdots$	c_m

where, $\sum_i a_{is} = \overline{c_i}$ holds.

2.2 Second Order Method

In this section, we construct the order condition for the second order method and present the method in Butcher table. To obtain the order condition for a second order method, we consider $p = 2$ in eqn. (7), which gives

$$b_1 + b_2 - 1 = 0$$

$$b_2\overline{c_2} + c_1 + c_2 - \frac{1}{2} = 0 \qquad (9)$$

as the required order condition with local truncation error

$$\frac{1}{2}b_2(\overline{c_2})^2 + c_2\overline{c_2} - \frac{1}{6(f_{xx} + 2f_{xy}f + f_yf^2)}) $$

$$+ \left(c_2\overline{c_2} - \frac{1}{6}\right)(f_xf_y + f_y^2 f)]h^3 + O(h^4) \qquad (10)$$

The order condition in (9) is two equations with five unknown parameters, which implies that there are three free parameters. Suppose the free parameters are $b_2, \overline{c_2}$ and c_2 then,

$$b_1 = 1 - b_2$$

$$c_1 = \frac{1}{2} - (b_2\overline{c_2} + c_2)$$

Domand and Prince, see[2], proposed that, having achieved a particular order of accuracy, the best strategy for practical purposes would be to choose the free parameters of an RK method of order p such that its error norm is minimize. The error norm is given by

$$A^{(p+1)} = \left\|\tau^{(p+1)}\right\|_2 = \sqrt{\sum_{i=1}^{n_{p+1}} (\tau_i^{(p+1)})^2} . \qquad (11)$$

$$\tau_1 = \frac{1}{2}b_2\overline{[(c)]_2}^2 + c_2\overline{c_2} - \frac{1}{6}$$

$$\tau_2 = c_2\overline{c_2} - \frac{1}{6}$$

which implies that

$$A^{(3)} = \left\|\tau^{(3)}\right\|_2 = \sqrt{\tau_1^2 + \tau_2^2} \qquad (12)$$

is the error norm of the proposed second order method. The following choice of free parameters minimizes the error norm and gives the best second order method.

$$b_2 = 0, \overline{c_2} = \frac{1}{3} \text{ and } c_2 = \frac{1}{2} \text{ then,}$$

$$A^{(3)} = \sqrt{0^2 + 0^2} = 0 ,$$

and the method is given in the Table 2.
with local truncation error

$$t_1(n + 1) = 0 + h^4/72(7f_{,xy} f_{,y} f + 6f_{,yy} f_{,y} f^2 +$$
$$5f_{,x} (f_{,yy} f + f_{,xy}) + 3(f_{,xyy} f^2 + f_{,xxy} f +$$
$$f_{,y}^2 f_{,x} + f_{,y}^3 f) + f_{,xxx} + f_{,y} f_{,xx} + f_{,yyy} f^3) + O(h^5) \qquad (13)$$

Table 2. Coefficients of the second order method

$\overline{c_i}$	a_{ij}		
0	0		
1/3	1/3	0	
b_i	1	0	
c_i	0	1/2	

Table 3. Coefficients of the third order method

$\overline{c_i}$	a_{ij}		
0	0		
1/2	1/2	0	
1/2	1/4	1/4	0
b_i	1	0	0
c_i	1/6	0	1/3

2.3 Third Order Method

In this section, we construct the order condition for the third order method and present the method in Butcher table. The order condition of the third order method is obtained by putting $p = 3$ in eqn. (7), which gives the following as the order condition for the third order methods

$$b_1 + b_2 + b_3 = 1$$

$$b_2\overline{c_2} + b_3\overline{c_3} + c_1 + c_2 + c_3 = \frac{1}{2}$$

$$\frac{1}{2}b_2(\overline{c_2})^2 + \frac{1}{2}b_3(\overline{c_3})^2 + c_2\overline{c_2} + c_3\overline{c_3} = \frac{1}{6}$$

$$\frac{1}{2}b_2(\overline{c_2})^2 + \frac{1}{2}b_3(\overline{c_3})^2 + c_2\overline{c_2} + c_3\overline{c_3} = \frac{1}{6} \quad (14)$$

$$c_2\overline{c_2} + c_3\overline{c_3} + b_3\overline{c_2}a_{32} = \frac{1}{6}$$

The third order condition in (14) consists of four equations with eight unknown parameters, which implies that there are four free parameters. We choose b_2, b_3, c_2 and c_3 as the free parameters and solved (14) in terms of the free parameters using maple package. We choose the values of the free parameters that minimize the error norm for the third order method according to the eqn. (11) and present in the table.

with local truncation error,

$$t_{n+1} = 0 + \frac{h^5}{1440}(2f_{xxxx} + 24f_{xyy}f_x f +$$
$$30f_{xyy}f_y f^2 + 9f_{xy}f_{yy}f^2 + 24f_{xy}f_y f_x +$$
$$18f_y^2 ff_{xy} + f_y^3 f + 12f_{yyy}f_x f^2 +$$
$$14f_{yyy}f_y f^3 + 3ff_{yy}f_{xx} + 27f_{yy}f_y^2 f^2 +$$
$$18f_{xxy}f_y f + 2f_{yyyy}f^4 + 3f_{xy}f_{xx} - 3f_y^2 f_{xx} +$$
$$6f_{xy}^2 f + 8f_{xyyy}f^3 + 6f_{yy}f_x^2 + 2f_y f_{xxx} +$$
$$12f_y^3 f_x + 3f_{yy}^2 f^3 + 8f_{xxxy}f + 12f_{xxy}f_x +$$
$$12f_y^4 f + 12f_{xxyy}f^2 + 36ff_{yy}f_y f_x) + O(h^6)$$

2.4 Fourth Order Method

To obtain the coefficients of the forth order method it is enough to construct the order condition of the method by putting $p = 4$ in eqn. (7). The following set of equations is the order condition which must be satisfied by the proposed forth order method.

$$b_1 + b_2 + b_3 + b_4 = 1$$

$$b_2\overline{c_2} + b_3\overline{c_3} + b_4\overline{c_4} + c_1 + c_2 + c_3 + c_4 = \frac{1}{2}$$

$$\frac{1}{2}b_2(\overline{c_2})^2 + \frac{1}{2}b_3(\overline{c_3})^2 + \frac{1}{2}b_4(\overline{c_4})^2 + c_2\overline{c_2} + c_3\overline{c_3} + c_4\overline{c_4} = \frac{1}{6}$$

$$c_2\overline{c_2} + c_3\overline{c_3} + c_4\overline{c_4} + b_3\overline{c_2}a_{32} + b_4(a_{42}\overline{c_2} + a_{43}\overline{c_3}) = \frac{1}{6}$$

$$b_2(\overline{c_2})^2 + b_3(\overline{c_3})^2 + b_4(\overline{c_4})^2 + b_3\overline{c_3}a_{32}c_2 + b_4\overline{c_4}(a_{42}\overline{c_2} + a_{43}\overline{c_3})$$
$$+ c_3a_{32}\overline{c_2} + c_4(a_{42}\overline{c_2} + a_{43}\overline{c_3}) = \frac{1}{8}$$

$$\frac{1}{2}c_2\overline{[(c]_2})^2 + \frac{1}{2}c_3(\overline{c_3})^2 + \frac{1}{2}c_4(\overline{c_4})^2 + \frac{1}{2}b_3a_{32}(\overline{c_2})^2 +$$
$$\frac{1}{2}b_4(a_{42}(\overline{c_2})^2 + a_{43}(\overline{c_3})^2) = \frac{1}{24}$$

$$b_4a_{43}a_{32}\overline{c_2} + c_3a_{32}\overline{c_2} + c_4(a_{42}\overline{c_2} + a_{43}\overline{c_3}) = \frac{1}{24}$$

Table 4. Coefficients of the forth order method

$\overline{c_i}$	a_{ij}			
0	0			
1/2	1/2	0		
1	0	1	0	
2/5	7/25	2/25	1/25	0
b_i	1	0	0	0
c_i	1/8	0	1/36	25/72

with local truncation

$$t_1(n+1) = -h^15/720(f_1y^12 + f_1xx \square + ff\square_1yy)(4f_1yy\, f^12 + 8ff_1xy -$$
$$f_1y^12 - f_1y^1\, f_1x + 4f_1xx) + O(h^16)$$

3. Stability of the Method

The stability of Runge-Kutta method of any stage is assessed by its stability polynomial[2]. In this section, we derive the stability polynomials of the second, third and fourth order methods developed in this paper.

3.1 Second Order Method

To obtain the stability polynomial of the second order method we solve the scalar test equation

$$y' = \lambda y \quad (18)$$

using the second order method, which gives the following:

$$k_{11} = f(x_n, y_n) = \lambda y_n$$

$$k_{21} = f(x_n + \overline{c_2}h, y_n + ha_{21}y_n) = \lambda(y_n + h\lambda a_{21}y_n)$$

$$y_{n+1} = y_n + h\lambda b_1 y_n + h\lambda b_2(y_n + h\lambda a_{21}y_n) +$$
$$h^2\lambda^2 c_1 y_n + h^2\lambda^2 c_2(y_n + h\lambda a_{21}y_n)$$

$$y_{n+1} = y_n(1 + h\lambda b_1 + h\lambda b_2(1 + h\lambda a_{21}) +$$
$$h^2\lambda^2 c_1 + h^2\lambda^2 c_2(1 + h\lambda a_{21}))$$

$$r(\mu) = 1 + \mu(b_1 + b_2) + \mu^2(c_1 + c_2 + a_{21}b_2) + \mu^3 a_{21}c_2 \quad (19)$$

where, $\mu = h\lambda$. Taking the values of the parameters from Table 2 into (19) we get

$$r(\mu) = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3 \quad (20)$$

which is the required stability polynomial of the proposed second order method. And the method is said to be absolutely stable if $|r(\mu)| < 1$.

3.2 Second Order Method

Applying the third order method on (18) also gives

$$k_{11} = \lambda y_n$$
$$k_{21} = \lambda [(y)]_n + h\lambda a_{21}y_n)$$
$$k_131 = \lambda \Box (y\Box_1 n + h\lambda \Box (a\Box_131\, k_111 + a_132\, k_121\, y_1 n)$$
$$k_{12} = \lambda^2 y_n$$
$$k_{22} = \lambda^2 [(y)]_n + h\lambda a_{21}y_n)$$

$$k_{31} = \lambda^2(y_n + h\lambda(a_{31}k_{11} + a_{32}k_{21}y_n) \,-\,$$

$$y_{n+1} = y_n + h\lambda b_1 y_n + b\lambda b_2(y_n + h\lambda a_{21}y_n)$$
$$+ h\lambda b_3(y_n + h\lambda(a_{31}\lambda y_n + a_{32}(y_n + h\lambda a_{21}y_n))) +$$
$$+ h^2\lambda^2(c_1 y_n + c_2 y_n + h\lambda a_{21}y_n) + h^2\lambda^2 c_3(y_n +$$
$$+ h\lambda(a_{31}\lambda y_n + a_{32}(y_n + h\lambda a_{21}y_n)))$$

Using $\mu = h\lambda$ and the values of parameters in Table 3 we obtain

$$r(\mu) = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3 + \frac{1}{24}\mu^4 \qquad (21)$$

Similarly, we obtained

$$r(\mu) = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3 + \frac{1}{24}\mu^4 + \frac{1}{144}\mu^5 \qquad (22)$$

which is the stability polynomial for forth order method.

4. Numerical Results

In this section, we conduct a numerical experiment by applying the proposed methods on some standard test problems and compare the results with the results of the existing methods in the literature. The following are the set of test problems:

Problem 1

$$y' = \frac{y}{4} - \frac{y^2}{80}, \qquad y(0) = 1$$

Exact solution: $y(x) = \dfrac{20}{1 + 19\exp\left(-\dfrac{x}{4}\right)}$

Problem 2

$$y' = y\cos(x), y(0) = 1,$$

Exact solution: $y(x) = \exp(\sin(x))$

Problem 3

$$y' = -y, y(0) = 1,$$

Exact solution: $y(x) = \exp(-x)$

Problem 4

$$y' = y + x^2 + 1, y(0) = \frac{1}{2}$$

Exact solution: $y(x) = (x+1)^2$
Source: [2,6,9].

We use XRK(P), RK(P) and WRK(P) to denote extended Runge-Kutta-like method developed in this paper, classical Runge-Kutta method Xinyuan method developed in[6] respectively, where 'P' is the order of the methods. The maximum global error of the methods taken for the values of $x \in [0,10]$ is given by
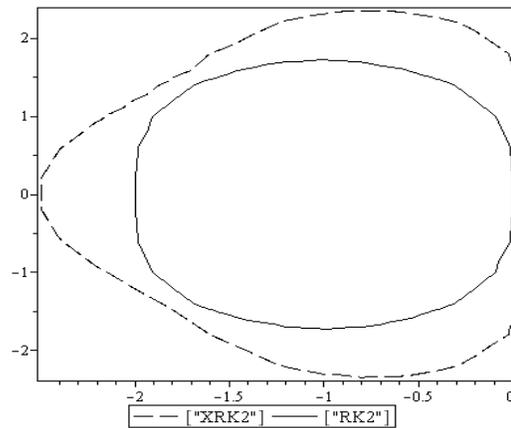
$$MXE = max\{|y(x_n) - y_n|\}.$$

Figure 1. Comparison of stability regions of second order methods.

Table 5. Comparison of maximum global error for second order methods

Problem	h	MXE	
		XRK2	RK2
1	0.1	6.49E-07	5.66E-04

	0.05	8.16E-08	1.40E-04
	0.025	1.02E-08	3.51E-05
	0.0125	1.28E-09	8.80E-06
2	0.1	4.37E-04	1.33E-03
	0.05	5.46E-05	3.11E-04
	0.025	6.82E-06	7.51E-05
	0.0125	8.52E-07	1.84E-05
3	0.1	1.66E-05	6.61E-04
	0.05	1.99E-06	1.59E-04
	0.025	2.44E-07	3.90E-05
	0.0125	3.02E-08	9.67E-06
4	0.1	3.07E+00	1.00E+02
	0.05	4.40E-01	2.63E+01
	0.025	5.14E-02	6.73E+00
	0.0125	6.50E-03	1.70E+00

Table 6. Comparison of maximum global error for third order methods

Problem	h	MXE		
		XRK3	RK3	WRK3
1	0.1	2.63E-09	2.59E-06	6.49E-07
	0.05	1.67E-10	3.27E-07	8.16E-08
	0.025	1.05E-11	4.10E-08	1.02E-08
	0.0125	6.58E-13	5.13E-09	1.28E-09
2	0.1	2.36E-06	1.96E-04	
	0.05	1.53E-07	2.44E-05	
	0.025	9.71E-09	3.04E-06	
	0.0125	6.12E-10	3.80E-07	
3	0.1	3.33E-07	1.66E-05	1.66E-05
	0.05	2.00E-08	1.99E-06	1.99E-06
	0.025	1.22E-09	2.44E-07	2.44E-07
	0.0125	9.56E-11	3.02E-08	3.02E-09
4	0.1	1.41E-02	2.49E+0	
	0.05	2.71E-03	3.27E-01	
	0.025	1.74E-04	4.20E-02	
	0.0125	1.10E-05	5.31E-03	

Table 7. Comparison of maximum global error for forth order methods

Problem	h	MXE		
		XRK4	RK4	WRK4
1	0.1	1.72E-09	1.63E-08	3.08E-06

	0.05	108E-10	1.03E-09	1.98E-07
	0.025	6.83E-12	6.47E-11	1.25E-08
	0.0125	4.28E-13	4.05E-12	7.88E-10
2	0.1	6.41E-07	1.29E-06	
	0.05	3.91E-08	7.53E-08	
	0.025	2.41E-09	4.53E-09	
	0.0125	1.51E-10	2.78E-10	
3	0.1	5.09E-08	3.33E-07	3.33E-07
	0.05	3.19E-09	2.00E-08	1.99E-08
	0.025	2.00E-10	1.22E-09	1.22E-09
	0.0125	1.25E-11	7.56E-10	7.56E-10
4	0.1	1.46E-02	4.08E-02	
	0.05	9.33E-04	2.71E-03	
	0.025	5.90E-05	1.74E-04	
	0.0125	3.71E-06	1.10E-05	
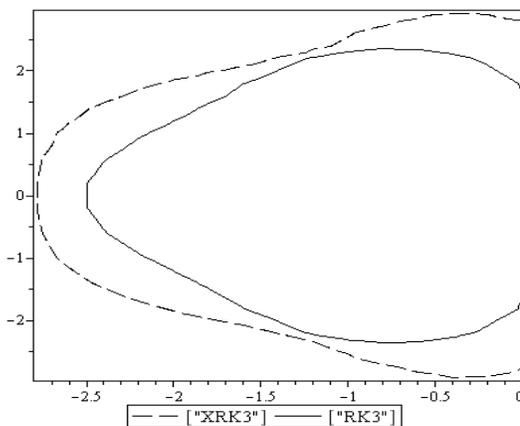
Figure 2. Comparison of stability regions of third order methods.
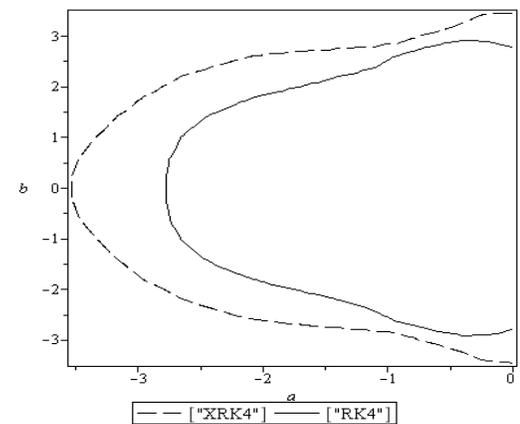
Figure 3. Comparison of stability regions of forth order methods

5. Conclusion

The family of improved extended Runge-Kutta-like methods which used the derivative of the function or equation itself has been proposed and a family of second, third and fourth order methods have been presented. The methods were tested on some standard problems and the results presented in terms of the maximum global error. The absolute stabilities of the methods are also considered.

The author in[6] did not derive the second order method, hence for the second order method we just compare the result only with the classical second order method (RK2).

For the third and fourth order methods we compared the results with the methods in[6] as well as the classical RK3 and RK4 method. However the methods in[6] are only suitable for autonomous ODEs, hence they cannot be used to solve non-autonomous ODEs (problems 2 and 4).

From the results in Tables 5–7 we can conclude that the proposed methods in this paper are better than classical Runge-Kutta methods and the methods developed in[6] of equal order in terms of computational accuracy.

Figures 1–3 also showed that the proposed methods have larger stability regions compared to classical Runge-Kutta methods as well as the methods in[6]. Hence, the proposed methods are more stable than the classical RK methods and the methods proposed in[6].

6. References

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