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A New Algorithm based on Shifted Legendre Polynomials for Fractional Partial Differential Equations

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Abstract

The aim of this article is to present a general framework of operational matrix via Shifted Legendre Polynomials (SLOM) for the solution of Fractional Partial Differential Equations (FPDEs) with variable coefficients. A new operational matrix is developed to approximate the solutions of two dimensional FPDEs. By using this operational matrix, a system of algebraic equations is attained. To demonstrate the efficiency and reliability of the proposed method, some numerical examples are conferred and compared with other existing approaches.

Keywords: Fractional Calculus, Fractional Partial Differential Equations, Shifted Legendre Polynomials

1. Introduction

In the recent years, the fractional calculus has been extensively utilized for the modeling of physical systems. During the past three decades, the topic of fractional calculus has attained the considerable attention of many researchers because of its wide range of usages in science and engineering. For the solution of differential and integral equations, it stipulates numerous potentially beneficial tools. To solve the time FPDEs is an important task. When taking a long-time scaling limit, it has been observed that the derivatives of fractional order with respect to time has generally arises as imperceptible generators of the time evolution. The approximated and numerical methods have been employed to tackle most of the Fractional Differential Equations (FDEs) because many of them do not have the exact solutions. Many researchers have been devoted their attentions towards the numerical and exact solutions of FDEs and FPDEs due to a wide range of applications. The analytical solutions of FDEs are still in an initial stage. To find the analytical and numerical solutions of these equations is a challenging task except few of them. In this regard, a number of endeavors have been taken into account to establish the techniques which can be used to obtain the analytical as well as numerical solutions of FDEs.

Many efforts have been attempted to develop different schemes which are used to solve FPDEs. To attain the approximate solutions of FPDEs, many numerical methods are well-known. Some of them are Homotopy Perturbation Method (HPM)[1], Reduced Differential Transform Method (RDTM)[2], Variational Iteration Method (VIM)[3], New Iterative Method (NIM)[4], Generalized Differential Transform Method (GDTM)[5], Jacobi tau approximation method[6] and wavelet method[7–10].

To solve the following FPDEs, we have developed a new algorithm:

$$\frac{\partial^{\eta} u(x,t)}{\partial t^{\eta}} + d_0(x,t)u(x,t) + d_1(x,t)\ \frac{\partial u(x,t)}{\partial x} + d_2(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + \cdots + d_n(x,t)\frac{\partial^n u(x,t)}{\partial x^n} = f(x,t), \qquad t > 0, \qquad x \epsilon R, \qquad (1)$$

with respect to the initial conditions:

$$u(x,0) = g_1(x), \qquad 0 < \alpha \le 1, \qquad t > 0$$

and

$$u(x,0) = g_2(x), \qquad u_t(x,0) = g_3(x), \qquad 1 < \alpha \le 2, \qquad t > 0,$$

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such that $d_i(i = 0,1,\ldots,n), g_1(x), \quad g_2(x), g_3(x)$ are known, $f(x,t)$ is the given source term, $,\eta$ is a parameter that interprets the order of the time-fractional derivative and $u(x,t)$ is the unknown.

For the solution of FPDEs, a number of methods have been employed such as Jacobi tau approximation which has been used to tackle space fractional diffusion equation[11]. The two dimensional block pulse functions along with the operational matrices of integration of fractional order have been investigated to get the solution of FPDES[12]. Likewise, in[13], the authors applied the homotopy perturbation method combined with the Laplace transform method to obtain the numerical solution of the fractional partial differential equations. Recently, many authors put their efforts to solve FPDEs by using the operational matrix approach[14–16]. These operational matrices are based on various orthogonal polynomials and wavelets. The results show that this method is really very simple and accurate.

In the presented article, the generalize form of the operational matrix technique for the solution of FPDEs with variable coefficients along with some initial conditions is discussed. The shifted Legendre polynomials are the base of this scheme. A new operational matrix has been established by using the properties of shifted Legendre polynomials. We analyzed from the numerical test that the method is eminently capable for solving such problems. The rest of the paper is organized as follows: In Section 2 some basic definitions of fractional calculus are given. In Section 3, a new idea is presented for the solution of a generalized class of FPDEs with variable coefficients. In Section 4, some numerical tests are performed to show the efficiency of the new technique. In Section 5, the conclusion of the paper is presented.

2. Preliminaries

Few of the basics definitions are given below:

Definition I[17,18]

The Riemann-Liouville (R-L) fractional derivative is defined as:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\dfrac{d^n}{dt^n}\displaystyle\int_0^t \frac{f(\tau)(d\tau)}{(t-\tau)^{\alpha-n+1}}, & n-1<\alpha<n \\ \dfrac{d^n}{dt^n}f(t), & n=\alpha \end{cases} \tag{2}$$

Definition II[17,18]

The Riemann- Liouville (R-L) fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu, \quad \mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \qquad \alpha > 0, \qquad t > 0. \tag{3}$$

Some Properties:

Some properties of Riemann-Liouville fractional integral are:

$$J^0 f(t) = f(t) \tag{4}$$

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \tag{5}$$

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t), \tag{6}$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{(\alpha+\gamma+1)} x^{\alpha+\gamma}. \tag{7}$$

Definition III[17,18]

The fractional derivative in Caputo sense is defined as:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t \frac{f^{(n)}(\tau)(d\tau)}{(t-\tau)^{\alpha-n+1}}, & n-1<\alpha<n \\ \dfrac{d^n}{dt^n}f(t), & n=\alpha \end{cases}, \text{where } n \in Z. \tag{8}$$

2.1 Some Properties

Some properties of fractional derivative in Caputo sense are:

$$D_t^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \tag{9}$$

$$J^\alpha D_x^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{k!} x^k, \qquad x > 0, \tag{10}$$

$$D^\alpha(\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t), \tag{11}$$

$$\frac{d^n}{dt^n}(\emptyset(t)f(t)) = \sum_{k=0}^{n} \binom{n}{k} \emptyset^k(t) f^{n-k}(t). \tag{12}$$

2.2 Shifted Legendre Polynomials

$$L_{k+1}(z) = \frac{2k+1}{k+1} z L_k(z) - \frac{k}{k+1} L_{k-1}(z), \qquad k = 1,2,\ldots \tag{13}$$

where

$$L_0(z) = 0, \qquad L_1(z) = z.$$

To transforms the interval $[-1,1]$ to $[0,1]$, one can use the transformation $t = \frac{(z+1)}{2}$ and the shifted Legendre polynomials are listed as

$$P_k(x) = \sum_{l=0}^{k} (-1)^{k+l} \frac{(k+l)!}{(k-l)!\,(l!)^2}, \qquad k = 0,1,2\ldots, \tag{14}$$

where $P_k(0) = (-1)^k, P_k(1) = 1$. The condition for orthogonality is defined as:

$$\int_0^1 P_k(x) P_h(x) dx = \begin{cases} \frac{1}{2k+1} & if\ k = h, \\ 0 & if\ k \neq h, \end{cases} \tag{15}$$

Which implies that any $f(x) \in C[0,1]$ can be approximate by Legendre polynomials as follows:

$$f(x) \approx \sum_{n=0}^{m} C_n P_n(x), \tag{16}$$

where

$$C_n = \langle f(x), P_n(x) \rangle = (2n+1) \int_0^1 f(x) P_n(x) dx.$$

$$f(x) = L_M^T \hat{P}_M, \tag{17}$$

Where $M = m+1, L$ depicts the coefficient vector and $\hat{P}$ is M terms vector function. The notion can be extended to two dimensional space and the two dimensional Legendre polynomials of order M as a product function of two

Legendre polynomials can be defined as:

$$P_t(x,y) = P_a(x) P_b(y), \qquad t = Ma + b + 1, \qquad a = 0,1,2,\ldots,m, b = 0,1,2,\ldots,m.$$

The orthogonality condition of $P_t(x,y)$ is given by:

$$\int_0^1 \int_0^1 P_n(x) P_s(y) P_c(x) P_d(y) dxdy = \begin{cases} \frac{1}{(2n+1)(2s+1)} & if\ n = c, s = d \\ 0 & otherwise. \end{cases} \tag{18}$$

By using the polynomial $P_t(x,y)$, consider any function $f(x,y) \in C([0,1] \times [0,1])$ can be approximated as follows:

$$f(x,y) \approx \sum_{n=0}^{m} \sum_{s=0}^{m} C_{ns} P_n(x) P_s(y). \tag{19}$$

Here,

$$C_{ns} = (2n+1)(2s+1) \int_0^1 \int_0^1 f(x,y) P_n(x) P_s(y) dxdy.$$

For convenience, use the notation $C_t = C_{ns}$ where $t = Mn + s + 1$ and in vector notation (6) can be written as:

$$f(x,y) \approx \sum_{t=1}^{M^2} C_t P_t(x,y) = L_{M^2}^T \psi(x,y), \tag{20}$$

Where L_{M^2} is the $1 \times M^2$ coefficient row vector and $\psi(x,y)$ is the $M^2 \times 1$ column vector of functions defined by, where

$$\psi(x,y) = (\psi_{11}(x,y) \ldots \psi_{1M}(x,y) \psi_{21}(x,y) \ldots \psi_{MM}(x,y))$$

$$\psi_{k+1,h+1}(x,y) = P_k(x) P_h(y),$$

for $k, h = 0,1,2,\ldots,m.$

2.3 Error Estimation

Any function $f(x,y)$ which is sufficiently smooth on $[0,1] \times [0,1]$, the error of the approximation is given by:

$$\|f(x,y) - P_n(x,y)\|_2 \leq \left(C_1 + C_2 + C_3 \frac{1}{M^{M+1}}\right) \frac{1}{M^{M+1}},$$

$$C_1 = \frac{1}{4} \max_{(x,y)\in[0,1]\times[0,1]} \left| \frac{\partial^{M+1}}{\partial x^{M+1}} f(x,y) \right|,$$

$$C_2 = \frac{1}{4} \max_{(x,y)\in[0,1]\times[0,1]} \left| \frac{\partial^{M+1}}{\partial y^{M+1}} f(x,y) \right|,$$

$$C_3 = \frac{1}{16} \max_{(x,y)\in[0,1]\times[0,1]} \left| \frac{\partial^{2M+2}}{\partial x^{M+1} \partial y^{M+1}} f(x,y), \right|.$$

Proof: The proof of the above mentioned theorem can be found in[21].

3. Generalized Fractional Differential Equation for Variable Coefficient

$$\frac{\partial^\eta u(x,t)}{\partial t^\eta} + d_0(x,t) u(x,t) + d_1(x,t) \frac{\partial u(x,t)}{\partial x} + d_2(x,t) \frac{\partial^2 u(x,t)}{\partial x^2} + \cdots + d_n(x,t) \frac{\partial^n u(x,t)}{\partial x^n} = f(x,t), \qquad t > 0, \qquad x \in R, \tag{21}$$

Constrained to the initial conditions:

$$u(x,0) = g_1(x), \qquad 0 < \alpha \leq 1, \qquad t > 0$$

and

$u(x,0) = g_2(x), \qquad u_t(x,0) = g_3(x), \qquad 1 < \alpha \le 2, \qquad t > 0,$

Where $d_i (i = 0,1,...,n)$, $g_1(x)$, $g_2(x)$, $g_3(x)$ are the known, $f(x,t)$ is the given source term, , η is a parameter that describes the order of the time-fractional derivative and $u(x,t)$ is the unknown. In terms of shifted Legendre series, we can write the solution of the above problem as:

$$\frac{\partial^\eta u(x,t)}{\partial t^\eta} = K\psi(x,t) \tag{22}$$

By applying the fractional integral of order η with respect to the initial conditions, we have:

$$u = KP^{\eta,t}\,\psi(x,t) + \sum_{k=0}^{n} t^k d_k, \tag{23}$$

Where $P^{\eta,t}$ represents the operational matrix of integration.

By simplifying the above equation, we get:

$$d_i \frac{\partial^i u(x,t)}{\partial x^i} = KP^{\eta,t}\, G^{(a_i,\beta_i,\alpha,\beta,\eta)}\psi + FG^{(a_i,\beta_i,\alpha,\beta,\eta)}\psi, \qquad i = 0,1,2,... \tag{24}$$

Where $G^{(a_i,\beta_i,\alpha,\beta,\eta)}$ is the operational matrix of derivative for variable coefficients:

Functional approximation for the source term can be written as:

$$f(x,t) = F_1\psi \tag{25}$$

By putting all the values in Equation (21), we have

$$K\psi = KP^{\eta,t}\, G^{(a_1,\beta_1,\alpha,\beta,\eta)}\psi + FG^{(a_1,\beta_1,\alpha,\beta,\eta)}\psi + KP^{\eta,t}\, G^{(a_2,\beta_2,\alpha,\beta,\eta)}\psi + FG^{(a_2,\beta_2,\alpha,\beta,\eta)}\psi + \cdots + KP^{\eta,t}\, G^{(a_n,\beta_n,\alpha,\beta,\eta)}\psi + FG^{(a_n,\beta_n,\alpha,\beta,\eta)}\psi + F_1\psi$$

After simplifying, we have:

$$(K - KP^{n,t}\, G^{(a_1,\beta_1,\alpha,\beta,\eta)} - KP^{n,t}\, G^{(a_2,\beta_2,\alpha,\beta,\eta)} - \cdots - KP^{n,t}\, G^{(a_n,\beta_n,\alpha,\beta,\eta)} + F(G^{(a_1,\beta_1,\alpha,\beta,\eta)} + G^{(a_2,\beta_2,\alpha,\beta,\eta)} + \cdots + G^{(a_n,\beta_n,\alpha,\beta,\eta)} - F_1)\psi = 0$$

$$(K - KP^{n,t}\, G^{(a_1,\beta_1,\alpha,\beta,\eta)} - KP^{n,t}\, G^{(a_2,\beta_2,\alpha,\beta,\eta)} - \cdots - KP^{n,t}\, G^{(a_n,\beta_n,\alpha,\beta,\eta)} + F(G^{(a_1,\beta_1,\alpha,\beta,\eta)} + G^{(a_2,\beta_2,\alpha,\beta,\eta)} + \cdots + G^{(a_n,\beta_n,\alpha,\beta,\eta)} - F_1) = 0. \tag{26}$$

4. Applications

Example 1

Consider the linear fractional partial differential Equation (**19**)

$$\frac{\partial^\alpha u}{\partial t^\alpha} + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 2(t^\alpha + x^2 + 1), 0 \le t \le 1, 0 \le x \le 1, 0 < \alpha \le 1 \tag{27}$$

Constrained to the initial condition,

$u(x,0) = x^2.$

The problem has the exact solution which is given as:

$$u(x,t) = x^2 + \frac{2t^{2\alpha}\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}.$$

Described methodology in Section 3 is used to solve the linear fractional partial differential problem by taking $m = 5$. Table 1 depicts the numerical results of current problem, which highlights the high accuracy of the proposed method.

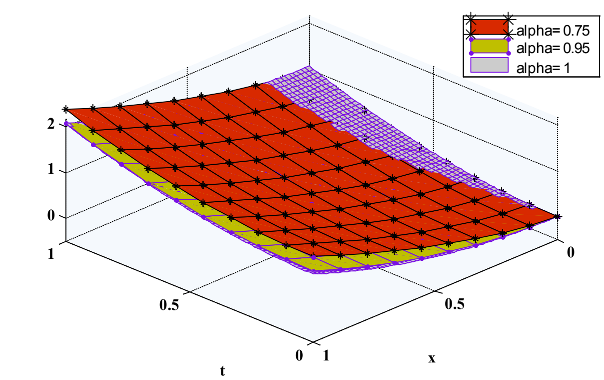

Figure 1(a). Approximate solution of example 1, at different choices of α .

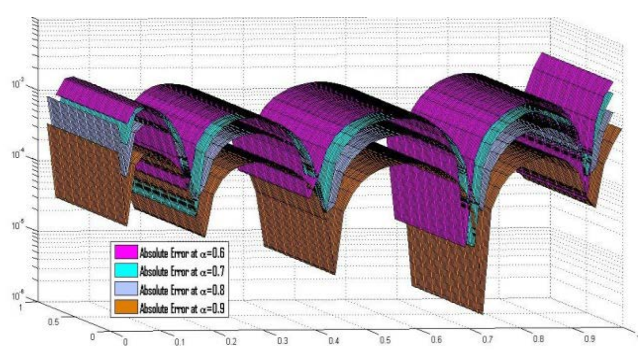

Figure 1(b). Absolute error of example 1, at different choices of α .

Table 1. Numerical comparison of example 1

t	x	SLOM	Sumudu Transform Method	SLOM	Sumudu Transform Method	SLOM	Sumudu Transform Method	Exact $\alpha = 0.75$	Exact $\alpha = 1$
		$\alpha = 0.75$		$\alpha = 0.95$		$\alpha = 1$			
0.2	0.2	0.163320	0.163679	0.090362	0.090389	0.08	0.08	0.16367554	0.08
	0.4	0.283320	0.283679	0.210362	0.210389	0.20	0.20	0.28367554	0.20
	0.7	0.613320	0.61368	0.540362	0.540389	0.53	0.53	0.61367554	0.53
0.5	0.2	0.529410	0.530488	0.327413	0.327369	0.29	0.290003	0.52887053	0.29
	0.4	0.649410	0.650675	0.447413	0.447370	0.41	0.410003	0.64887053	0.41
	0.7	0.979410	0.981188	0.777413	0.777373	0.74	0.740004	0.97887053	0.74
0.8	0.2	1.028680	1.06802	0.741783	0.742455	0.68	0.680197	1.02940439	0.68
	0.4	1.148680	1.19247	0.861783	0.862523	0.80	0.80022	1.14940439	0.80
	0.7	1.478680	1.53473	1.191783	1.192710	1.13	1.13028	1.47940439	1.13

Example 2

Consider the linear time-fractional wave Equation **[20]**

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{2} x^{2} \frac{\partial^{2} u}{\partial x^{2}}, t > 0, \qquad x \epsilon R, 1 < \alpha \leq 2, \qquad (28)$$

constrained to the initial conditions,

$$u(x, 0) = x, \qquad \frac{\partial u(x, 0)}{\partial t} = x^{2}.$$

The exact solution of the problem is given as:

$$u(x, t) = x + x^{2} \sinh t.$$

Described methodology in section 3 is used to solve the linear fractional partial differential problem by taking $m = 8$. Table 2 depicts the numerical results of current problem, which highlight the high accuracy of the proposed method.

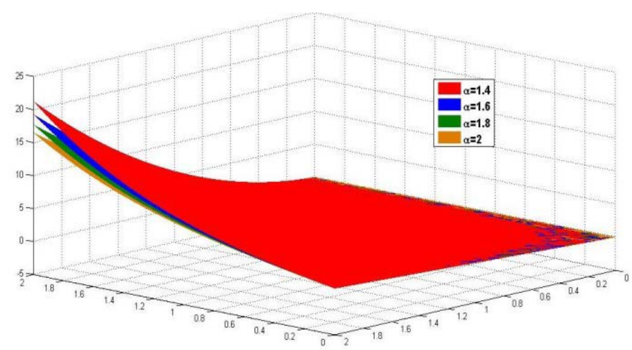

Figure 2(a). Approximate solution of example 2, for different choices of α.

Table 2. Numerical comparison of example 2

t	x	SLOM	ADM	VIM	SLOM	ADM	VIM	SLOM	Exact
		$\alpha = 1.5$			$\alpha = 1.75$			$\alpha = 2$	
0.2	0.25	0. 26284107	0.26284061	0.26269693	0. 26267007	0.26266989	0.26248505	0. 26258350	0.26258350
	0.50	0. 55136431	0.55136246	0.55078773	0.55068029	0.55067959	0.55059402	0. 55033400	0.55033400
	0.75	0. 86556971	0.8655655	0.86427239	0.86403066	0.86402909	0.86383654	0. 86325151	0.86325150
	1.0	1. 20545727	1.20544984	1.20315093	1.20272118	1.20271839	1.20237608	1. 20133602	1.20133360
0.4	0.25	0. 27697140	0.27697113	0.27642739	0.27615680	0.27615668	0.27607615	0. 27567201	0.27567202
	0.50	0. 60788563	0.60788455	0.60570958	0.60462721	0.60462675	0.60430459	0. 60268804	0.60268808
	0.75	0. 99274267	0.99274024	0.98784655	0.98541124	0.98541019	0.98468533	0. 98104811	0.98104818
	1.0	1. 43154252	1.43153821	1.42283824	1.41850887	1.41850702	1.41721837	1. 41075219	1.41075232
0.6	0.25	0. 29309445	0.29309481	0.29198616	0.29108982	0.29109009	0.29092233	0. 28979085	0.28979084
	0.50	0. 67237780	0.67237923	0.66794464	0.66435928	0.66436039	0.66368931	0. 65916343	0.65916339
	0.75	1.13785005	1.13785532	1.12787544	1.11980839	1.11981088	1.11830097	1. 10811772	1.10811764
	1.0	1.68951121	1.68951694	1.67177856	1.65743715	1.65744156	1.65475727	1. 63665373	1.63665358

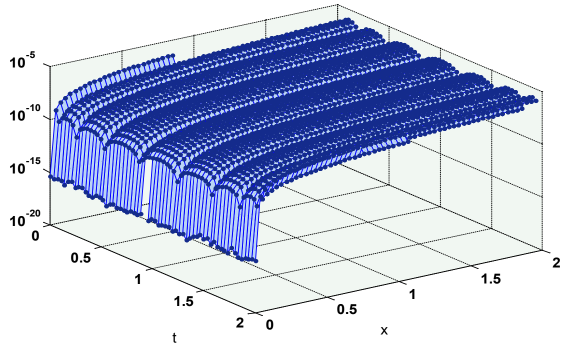

Figure 2(b). Absolute error of example 2, for different choices of α .

Example 3

Consider the linear time-fractional wave equation **[20]**

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{2} u}{\partial x^{2}}, t > 0, \qquad x \epsilon R, 0 < \alpha \leq 1, \qquad (29)$$

constrained to the initial condition,

$$u(x, \mathbf{0}) = sinx.$$

The exact solution of the problem is given only for the case when $\alpha = \mathbf{1}$, i.e $u(x,t) = e^{-t} \sin(x)$.

Described methodology in Section 3 is used to solve the linear fractional partial differential problem by taking $m = 8.$ Table 3 depicts the numerical results of current problem, which highlight the high accuracy of the proposed method.

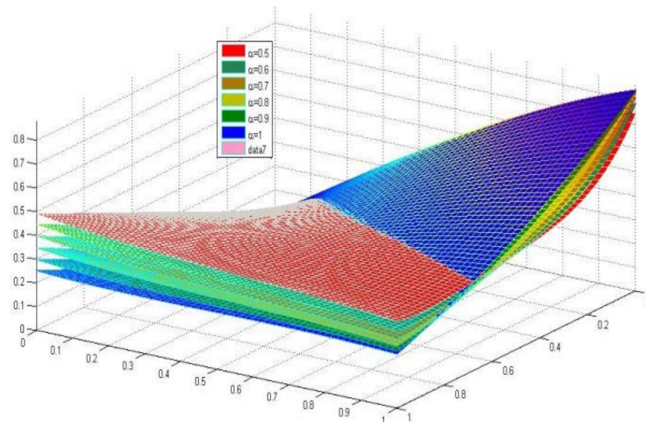

Figure 3(a). Approximate solution of example 3, for different choices of α .

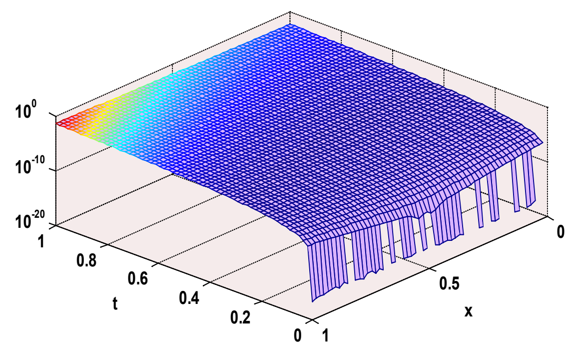

Figure 3(b). Absolute error of example 3, for different choices of α .

Table 3. Numerical comparison of example 3

t	x	SLOM	ADM	VIM	SLOM	ADM	VIM	SLOM	Exact
		$\alpha = 0.5$			$\alpha = 0.75$			$\alpha = 1$	
0.2	0.25	0. 15478583	0.15539194	0.15580311	0. 18084874	0.18094705	0.17925384	0. 20252818	0.20255723
	0.50	0. 30153999	0.30112237	0.30191914	0. 35069902	0.35064369	0.34736255	0. 39249464	0.39252043
	0.75	0. 42757427	0.42813046	0.42926330	0. 49844044	0.49853901	0.49387393	0. 55802007	0.55807861
	1.0	0. 52393251	0.52851947	0.52991795	0. 61467464	0.61543759	0.60967863	0. 68877368	0.68893817
0.4	0.25	0. 12050138	0.12272319	0.11364623	0. 14616004	0.14674143	0.14386345	0. 16548829	0.16583983
	0.50	0. 23802151	0.23781605	0.22022640	0. 28441044	0.28435919	0.27878393	0. 32093645	0.32136855
	0.75	0. 33515515	0.33122668	0.31311413	0. 40351939	0.40429688	0.39637006	0. 45612228	0.45691612
	1.0	0. 40287103	0.41740644	0.38653386	0. 49520709	0.49990973	0.48931183	0. 56241241	0.56405487
0.6	0.25	0. 08982510	0.09310943	0.10020628	0. 11844697	0.11963459	0.12470195	0. 13422302	0.13577817
	0.50	0. 18236498	0.18042976	0.19418223	0. 23247047	0.23183088	0.24165055	0. 26094739	0.26311431
	0.75	0. 25328469	0.25653185	0.27608465	0. 32839806	0.32961305	0.34357448	0. 37039953	0.37409128
	1.0	0. 29281629	0.31668403	0.34082188	0. 39821868	0.40690147	0.42413661	0. 45509900	0.45180906

5. Conclusion

The applications of shifted Legendre polynomials were extended successfully for solving linear fractional partial differential equations with variable coefficient. The comparison of the presented method with other numerical methods illustrated that our method can accurately represent properties of fractional calculus. The obtained results demonstrate the validity and applicability of proposed method for solving the FPDEs.

6. Acknowledgments

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