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The differential transform method for solving multidimensional partial differential equations

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Abstract
In this work, an analytical solution of linear and nonlinear multidimensional partial differential equation is deduced by the Differential Transform Method (DTM). Some numerical examples are presented to demonstrate the efficiency and reliability of this method.

Keywords: Differential transform method; Multidimensional partial differential equation.

Introduction

Solving partial differential equations (PDEs) is an absolute necessity in the context of Applied Mathematics, Theoretical Physics and Engineering Sciences. It is inevitable that PDEs will appear while conducting research in these areas. Therefore, it is vital to extract solutions for any given PDE by using some mathematical technique or so. This will lead to a lot of scientific information in the context of the above mentioned research areas. After all, a closed form solution of any given PDEs is a stepping stone towards further meaningful investigation into the problem. Therefore, it is important to venture into several techniques of integrability of these PDEs. One such method is the differential transform method.

The concept of differential transform method (DTM) was introduced first by Zhou (1986). This scheme is based on the taylor series expansion to construct analytical solutions in the form of a polynomial by means of an iterative procedure. Recently, researchers have applied the DTM to obtain analytical solutions for linear and nonlinear differential equations such as two point boundary value problem (Chenand & Liu, 1998), the KdV and MKdV equations (Angalgil & Ayaz, 2009), the nonlinear parabolic-hyperbolic partial differential equations (Biazar *et al.,* 2010), the two-dimensional nonlinear Gas dynamic and Klien-Gordon equations (Jafari *et al.,* 2010a).

In this paper, we are interested in extending the applicability of differential transform method to the three-dimensional nonlinear initial boundary value problem IBVP of the form equation:

$$u_{tt} = F(x, y, z, u, u^n_{xx}, u^n_{yy}, u^n_{zz}), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \quad n > 1$$

subject to the boundary conditions:

$$u(0, y, t) = f_1(y, t) \qquad u(a, y, t) = f_2(y, t),$$
$$u(x, 0, t) = f_3(x, t) \qquad u(x, b, t) = f_4(x, t),$$

and with the initial conditions:

$$u(x, y, 0) = f_5(x, y), \quad u_t(x, y, 0) = f_6(x, y).$$

Also we consider the differential transform method to solve the three-dimensional linear Helmholtz equation in the following form:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + c\frac{\partial^2 u}{\partial z^2} + \lambda u = F(x,y,z),$$

with the initial conditions:

$$u(0,y,z) = f_1(y,z) \quad u_x(0,y,z) = f_2(y,z),$$
$$u(x,0,z) = f_3(x,z) \quad u_y(x,0,z) = f_4(x,z),$$
$$u(x,y,0) = f_5(x,y) \quad u_z(x,y,0) = f_6(x,y),$$

where

$f_1(y,z),\ f_2(y,z),\ f_3(x,z),\ f_4(x,z),\ f_5(x,y),\ f_6(x,y)$, and a, b, c, λ are given functions and constant respectively.

These equations have been used in various fields such as engineering and physics. For more details about these equations the reader is referred to (Zwillinger, 1992; Burdenand & Faires, 1993). Jafari & Zabihi solved the above equations by homotopy perturbation method and homotopy analysis method respectively (Jafari *et al.,* 2010b & 2010c).We wants to apply the (DTM) for the linear Helmholtz equation and the nonlinear IBVP equation. Several numerical experiments of linear and nonlinear partial differential equations have presented.

Basic ideas of differential transform method

In this section, we want to demonstrate the basic definitions operations of the m- dimensional differential transform are defined in [1] as follows:

$$U(k_1,k_2,\dots,k_m) = \frac{1}{k_1!\,k_2!\dots k_m!}\left[\frac{\partial^{k_1+k_2+\dots+k_m} u(x_1,x_2,\dots,x_m)}{\partial x_1^{k_1}\dots\partial x_m^{k_m}}\right]_{(0,0,\dots,0)} \qquad (1)$$

Where $u(x_1, x_2, x_m)$ is the original and $U(k_1, k_2, \dots, k_m)$ is the transformed function. The inverse differential transform of $U(k_1, k_2, \dots, k_m)$ is defined as :

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$$u(x_1,x_2,\dots,x_m)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\dots\sum_{k_m=0}^{\infty}U(k_1,k_2,\dots,k_m)\,x_1^{k_1}x_2^{k_2}\dots x_m^{k_m}, \qquad (2)$$

Through Eqs.(1) and (2) the function $u(x_1,x_2,\dots,x_m)$ is expressed by a series in the following form:

$$u(x_1,\dots,x_m)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\dots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\dots k_m!}\left[\frac{\partial^{k_1+k_2+\dots+k_m}u(x_1,\dots,x_m)}{\partial x_1^{k_1}\dots\partial x_m^{k_m}}\right]_{(0,0,\dots,0)}x_1^{k_1}x_2^{k_2}\dots x_m^{k_m} \quad (3)$$

On the basis of the definitions Eqs. (1)-(3) we can easily prove the following theorems :

Theorem 1

If $u(x_1,x_2,\dots,x_m)=f_1(x_1,x_2,\dots,x_m)+f_2(x_1,x_2,\dots,x_m)$ then $U(k_1,k_2,\dots,k_m)=F_1(k_1,k_2,\dots,k_m)+F_2(k_1,k_2,\dots,k_m)$.

Theorem 2

If $u(x_1,x_2,\dots,x_m)=\lambda f(x_1,x_2,\dots,x_m)$ then

$$U(k_1,k_2,\dots,k_m)=\lambda F(k_1,k_2,\dots,k_m).$$

where, λ is a constant.

Theorem 3

If $u(x_1,x_2,\dots,x_m)=\frac{\partial f(x_1,x_2,\dots,x_m)}{\partial x_1}$ then

$$U(k_1,k_2,\dots,k_m)=(k_i+1)F(k_1,k_2,\dots,(k_i+1),\dots,k_m).$$

Theorem 4

If $u(x_1,x_2,\dots,x_m)=\frac{\partial^{(r+s)}f(x_1,x_2,\dots,x_m)}{\partial x_i^r\partial x_j^s}$, $1\le i\ne j\le m$ then

$$U(k_1,k_2,\dots,k_m)=(k_i+1)\dots(k_i+r)(k_j+1)\dots(k_j+s)F(k_1,\dots,(k_i+r),\dots,(k_j+s),\dots,k_m).$$

Theorem 5

If $u(x_1,x_2,\dots,x_m)=x_1^{h_1}x_2^{h_2}\dots x_m^{h_m}$ then

$$U(k_1,k_2,\dots,k_m)=\delta(k_1-h_1)\delta(k_2-h_2)\dots\delta(k_m-h_m)$$

, where

$$\delta(k_i-h_i)=\begin{cases}1 & k_i=h_i\\ 0 & \text{other wise}\end{cases}$$

Theorem 6

If $u(x_1,x_2,\dots,x_m)=f_1(x_1,x_2,\dots,x_m)\,f_2(x_1,x_2,\dots,x_m)$ then

$$U(k_1,k_2,\dots,k_m)=\sum_{r_1=0}^{k_1}\sum_{r_2=0}^{k_2}\dots\sum_{r_m=0}^{k_m}F_1(r_1,k_2-r_2,\dots,k_m-r_m)F_2(k_1-r_1,r_2,\dots,r_m).$$

Theorem 7

If $u(x_1,x_2,\dots,x_m)=x_1^{h_1}x_2^{h_2}\dots\sin(ax_i+b)\dots x_m^{h_m}$ then

$$U(k_1,k_2,\dots,k_m)=\delta(k_1-h_1)\delta(k_2-h_2)\dots\frac{a^{k_i}}{k_i!}\sin\left(\frac{k_i\pi}{2}+b\right)\dots\delta(k_m-h_m).$$

.

Theorem 8

If $u(x_1,x_2,\dots,x_m)=x_1^{h_1}x_2^{h_2}\dots\cos(ax_i+b)\dots x_m^{h_m}$ then

$$U(k_1,k_2,\dots,k_m)=\delta(k_1-h_1)\delta(k_2-h_2)\dots\frac{a^{k_i}}{k_i!}\cos\left(\frac{k_i\pi}{2}+b\right)\dots\delta(k_m-h_m).$$

Illustrative examples

For purposes of illustration of DTM for solving linear and nonlinear multidimensional partial differential equations, we present four examples

Example 1 Consider the following three-dimensional Helmholtz equation:

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}-\frac{\partial^2u}{\partial z^2}-4u=(12x^2-4x^4)\sin(y)\cos(z)$$

(4)

with the initial condition :

$$u(0,y,z)=0,\qquad u_x(0,y,z)=0 \qquad (5)$$

The exact solution can be expressed as:

$$u(x,t)=x^4\sin(y)\cos(z)$$

Taking the differential transform of Eq.(4), leads to

$$(k_1+2)(k_1+1)U(k_1+2,k_2,k_3)+(k_2+2)(k_2+1)U(k_1,k_2+2,k_3)$$
$$-(k_3+2)(k_3+1)U(k_1,k_2,k_3+2)-4U(k_1,k_2,k_3)=$$
$$12\delta(k_1-2)\frac{1}{k_2!}\sin\left(\frac{k_2\pi}{2}\right)\frac{1}{k_3!}\cos\left(\frac{k_3\pi}{2}\right)-4\delta(k_1-4)\frac{1}{k_2!}\sin\left(\frac{k_2\pi}{2}\right)\frac{1}{k_3!}\cos\left(\frac{k_3\pi}{2}\right) \qquad (6)$$

From the initial conditions given by equations Eq.(5)we have:

$$U(0,k_2,k_3)=0,$$
$$U(1,k_2,k_3)=0,\qquad k_2,k_3=0,1,2,\dots \qquad (7)$$

substituting equation (8) into (6) and by means of recursive method, the results are listed as follows:

$$U(k_1,k_2,k_3)=0\quad if\quad k_1\ne4\quad and\quad k_2,k_3=0,1,2,\dots$$

$$U(4,k_2,k_3)=\frac{1}{k_2!}\sin\left(\frac{k_2\pi}{2}\right)\frac{1}{k_3!}\cos\left(\frac{k_3\pi}{2}\right)\quad if\quad k_2,k_3=0,1,2,\dots$$

We obtained the series solution as

$$u(x,y,z)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}U(k_1,k_2,k_3)\,x^{k_1}y^{k_2}z^{k_3}=x^4\sin(y)\cos(z)$$

which is an exact solution of the problem.

Example 2 Consider the following two-dimensional Schrodinger equation:

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}-2u=(12x^2-3x^4)\sin(y) \qquad (8)$$

with the initial condition :

$$u(0,y)=0,\qquad u_x(0,y)=0 \qquad (9)$$

The exact solution can be expressed as :

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$$u(x,t) = x^4 \sin(y)$$

Taking the differential transform of Eq.(8) leads to

$$(k_1+2)(k_1+1)U(k_1+2,k_2)+(k_2+2)(k_2+1)U(k_1,k_2+2)-2U(k_1,k_2,k_3+2)$$
$$= 12\delta(k_1-2)\frac{1}{k_2!}\sin\left(\frac{k_2\pi}{2}\right)-3\delta(k_1-4)\frac{1}{k_2!}\sin\left(\frac{k_2\pi}{2}\right) \qquad (10)$$

The transformed version of Eq.(9) is :

$$U(0,k_2)=0,$$
$$U(1,k_2)=0, \qquad k_2,k_3=0,1,2,\ldots \qquad (11)$$

Substituting equation (11) into (10), all spectra can be found as

$$U(k_1,k_2)=\begin{cases}\frac{1}{k_2!}\sin\left(\frac{k_2\pi}{2}\right) & if \ k_1=4\\ 0 & other\ wise\end{cases}$$

Thus, we obtained:

$$u(x,y)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}U(k_1,k_2)\,x^{k_1}y^{k_2}=x^4\sin(y)$$

Which is the exact solution of Eq.(8).

Example 3 Consider the following two-dimensional nonlinear inhomogeneous (IBVP):

$$\frac{\partial^2 u}{\partial t^2}=2x^2+2y^2+\frac{15}{2}\left(x\frac{\partial^2 u^2}{\partial x^2}+y\frac{\partial^2 u^2}{\partial y^2}\right) \qquad (12)$$

$$u(0,y,t)=y^2t^2+yt^6 \qquad u(1,y,t)=(1+y^2)t^2+(1+y)t^6$$

$$u(x,0,t)=x^2t^2+xt^6 \qquad u(x,1,t)=(1+x^2)t^2+(1+x)t^6$$

and initial conditions :

$$u(x,y,0)=0, \qquad u_t(x,y,0)=0 \qquad (13)$$

The exact solution can be expressed as :

$$u(x,y,t)=(x^2+y^2)t^2+(x+y)t^6 .$$

Taking the differential transform of Eq.(12), leads to

$$(k_3+2)(k_3+1)U(k_1,k_2,k_3+2) = 2\delta(k_1-2)\delta(k_2)\delta(k_3)+2\delta(k_1)\delta(k_2-2)\delta(k_3)$$

$$+\frac{15}{2}\sum_{s=0}^{k_1}\delta(s-1)\sum_{i=0}^{k_1-s}\sum_{j=0}^{k_2}\sum_{k=0}^{k_3}(i+2)(i+1)(k_1-s-i+2)(k_1-s-i+1)$$

$$U(i+2,k_2-j,k_3-k)U(k_1-s-i+2,j,k)+\frac{15}{2}\sum_{s=0}^{k_2}\delta(s-1)\sum_{i=0}^{k_1}\sum_{j=0}^{k_2-s}\sum_{k=0}^{k_3}.$$

(j+2)(j+1)($k_2-s-j+2$)

$$(k_2-s-j+1)U(i,k_2-s-j+2,k_3-k)U(k_1-i,j+2,k) \qquad (14)$$

The transformed version of Eq.(13) is :

$$U(0,k_2,k_3)=0,$$
$$U(1,k_2,k_3)=0, \qquad k_2,k_3=0,1,2 \qquad (15)$$

Substituting equation (15) into (14), all spectra can be found as

$$U(k_1,k_2,k_3)=\begin{cases}1 & if\ k_2=0\ and\ k_1=k_3=2\\ 1 & if\ k_1=0\ and\ k_2=k_3=2\\ 1 & if\ k_1=1,\ k_2=0,\ k_3=6\\ 1 & if\ k_1=0,\ k_2=1,\ k_3=6\\ 0 & otherwise\end{cases}$$

Which we have:

$$u(x,y,t)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}U(k_1,k_2,k_3)x^{k_1}y^{k_2}t^{k_3}=(x^2+y^2)t^2+(x+y)t^6$$

Thus, we obtained the exact solution.

Example 4 Consider the following two-dimensional nonlinear nonhomogeneous partial differential equation:

$$\frac{\partial^2 u}{\partial t^2}=2e^t+u-\left(e^{-x}\left(\frac{\partial^2 u}{\partial x^2}\right)^2+e^{-y}\left(\frac{\partial^2 u}{\partial y^2}\right)^2\right), \quad 0<x,y<1, \quad t>0 \quad (16)$$

Subject to the Neumann boundary conditions

$$u_x(0,y,t)=1 \qquad u_x(1,y,t)=e$$
$$u_y(x,0,t)=1 \qquad u_y(x,1,t)=e$$

and initial conditions:

$$u(x,y,t)=e^x+e^y, \qquad u_t(x,y,0)=1 \qquad (17)$$

The exact solution can be expressed as :

$$u(x,y,t)=te^t+e^x+e^y$$

Taking the differential transform of Eq.(16), leads to

$$(k_3+2)(k_3+1)$$
$$U(k_1,k_2,k_3+2)=2\delta(k_1)\delta(k_2)\frac{1}{k_3!}+U(k_1,k_2,k_3)-$$
$$\sum_{s=0}^{k_1}\frac{(-1)^s}{s!}\sum_{i=0}^{k_1-s}\sum_{j=0}^{k_2}\sum_{k=0}^{k_3}(i+2)(i+1)(k_1-s-i+2)(k_1-s-i+1)$$

$$U(i+2,k_2-j,k_3-k)U(k_1-s-i+2,j,k)+\sum_{s=0}^{k_3}\frac{(-1)^s}{s!}\sum_{i=0}^{k_1}\sum_{j=0}^{k_2}\sum_{k=0}^{k_3}.$$

(j+2)(j+1)($k_2-s-j+2$)

$$(k_2-s-j+1)U(k_1-i,j+2,k)U(i,k_2-s-j+2,k_3-k) \qquad (18)$$

From the initial conditions given by equations Eqs.(17)we have:

$$U(0,0,0)=2$$

$$U(k_1,0,0)=\frac{1}{k_1!} \qquad k_1=1,2,\ldots$$

$$U(0,k_2,0)=\frac{1}{k_2!} \qquad k_2=1,2,\ldots$$

$$U(k_1,k_2,0)=0 \qquad else$$

$$U(0,0,1)=1$$

$$U(k_1,k_2,1)=0 \quad if \quad k_1\neq 0 \;\; and \;\; k_2\neq 0 \qquad (19)$$

Substituting equation (19) into (18) and by means of recursive method, the results
are listed as follows:

$$U(k_1,k_2,k_3)=\begin{cases} 2 & if\ k_1=k_2=k_3=0 \\ \frac{1}{k_1!} & if\ k_1\neq 0,\ k_2=k_3=0 \\ \frac{1}{k_2!} & if\ k_2\neq 1,\ k_1=k_3=0 \\ \frac{1}{(k_3-1)!} & if\ k_3\neq 0,\ k_1=k_2=0 \\ 0 & otherwise \end{cases}$$

Which we have:

$$u(x,y,t)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}U(k_1,k_2,k_3)x^{k_1}y^{k_2}t^{k_3}=te^t+e^x+e^y$$

Thus, we obtained the exact solution.

Conclusion

The DTM has been successfully applied to obtain the solution of linear and nonlinear multidimensional PDEs. The examples show that the results of the present method are in excellent agreement with the exact solutions. It is apparently seen that DTM is a very powerful and efficient technique in finding analytical solutions for wide classes of linear and nonlinear problems .In future, this method will be utilized to extract solutions of the vector coupled PDEs in multidimensions . Such coupled vector PDEs also appear in various areas of Physical, Chemical and Biological sciences. These results will be reported in future publications.

The *Mathematica* Package was used to calculate the series obtained by differential transform method.

References

1. Angalgil FK and Ayaz F (2009) Solitary wave solutions for the KdV and mKdV equations by differential transform method. *J. Chaos, Solitons & Fractals.* 41, 464-472.
2. Biazar J, Eslami M and Islam MR (2010) Differential Transform Method for Nonlinear Parabolic-hyperbolic Partial Differential Equations. *J. Appl & Appl. Math*. 5(10) 1493-1503.
3. Burdenand RL and Faires JD (1993) Numerical Analysis. PWS Publ. Co., Boston,
4. Chenand CL and Liu YC (1998) Solution of two point boundary value problems using the differential transformation method, *J. Opt. Theory Appl.* 99, 23-35.
5. Jafari H, Alipour M and Tajadodi H (2010a) Two-dimensional differential transform method for solving nonlinear partial differential equations. *J. Res. & Rev. Appl. Sci*. 63, 968-971.
6. Jafari H, Saeidy M and Firoozjaee MA (2010c) The homotopy analysis method for solving higher dimensional initial boundary value problems of variable. *Numer. Methods Partial Differ. Equations*. 26(5), 1021-1032.
7. Jafari H, Saeidy M and Zabihi M (2010b) Application of homotopy perturbation method to multidimensional partial differential equations. *Int. J. Computer Math.* 87(11), 2444-2449.
8. Zhou JK (1986) Differential transformation and its application for electrical circuits. Huarjung University Press, Wuuhahn, China, (in Chinese).
9. Zwillinger D (1992) Handbook of differential equations. Academic Press, Boston, M.A.