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# A note on the nonabelian tensor square

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**Abstract**

In this paper, we determine the nonabelian tensor square $G \otimes G$ for special orthogonal groups $SO_n(F_q)$ and spin groups $Spin_n(F_q)$, where $F_q$ is a field with $q$ elements.

**Keywords:** Special orthogonal group; Spin group; Nonabelian tensor square.

## Introduction

For a group $G$, the nonabelian tensor square $G \otimes G$ is the group generated by the symbols $g \otimes h$ and defined by the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h),$$
$$g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

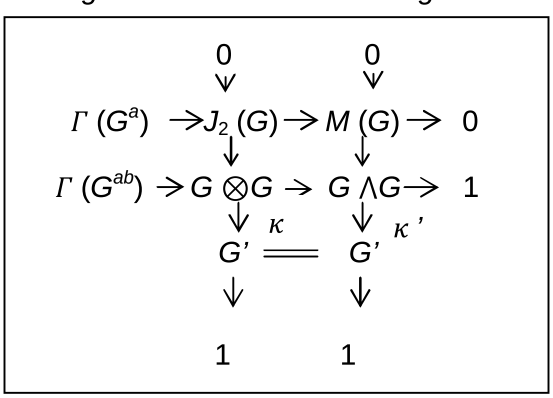
for all $g, g', h, h' \in G$, where ${}^{g}g' = gg'g^{-1}$. The nonabelian tensor square is a special case of the nonabelian tensor product which has its origin in homotopy theory and was introduced in 1984, 1987 (Brown & Loday 1984 & 1987). The exterior square $G \wedge G$ is obtained by imposing the additional relations $g \otimes g = 1_{\otimes}$ for all $g \in G$ on $G \otimes G$. The commutator map induces homomorphisms

$$: g \otimes h \in G \otimes G \longrightarrow \kappa(g \otimes h) = [g, h] \in G',$$

$$\kappa' : g \wedge h \in G \wedge G \longrightarrow \kappa'(g \wedge h) = [g, h] \in G'$$
and $J_2(G) = \ker(\kappa)$.

The results of Brown and Loday (1984 & 1987) give the following commutative diagram with exact rows and central extensions as columns:

*Fig. 1. The commutative diagram*




where $G'$ is the commutator subgroup of $G$, $M(G)$ the multiplicator of $G$ and $\Gamma$ the Whitehead's quadratic function (Whitehead, 1950).

The determination of $G \otimes G$ for linear groups was mentioned as an open problem by Brown *et al.* (1987) and was pointed out in a more general form by Kappe (1999). In the latter paper, there is a list of open problems on the computation of the nonabelian tensor square of finite groups. (Hannebauer, 1990) determined the nonabelian tensor square of $SL_2(F_q)$, $PSL_2(F_q)$, $GL_2(F_q)$ and $PGL_2(F_q)$ for all $q \geq 5$ and $q = 9$. Later, in 2008 (Erfanian *et al.*, 2008) determined the nonabelian tensor square of $SL_n(F_q)$, $PSL_n(F_q)$, $GL_n(F_q)$ and $PGL_n(F_q)$ for all $n, q \geq 2$. The Schur multiplier and nonabelian tensor square of special linear groups, projective special linear groups, symplectic groups and projective symplectic groups determined by in 2011 (Rashid *et al.,* 2011a). They also computed the nonabelian tensor square of groups of order $p^2q$ (Rashid *et al.,* 2011b).

In this paper, we focus on the Schur multiplier and nonabelian tensor square of special orthogonal groups $SO_n(F_q)$ and spin groups $Spin_n(F_q)$, where $F_q$ is a field with $q$ elements.

The nonabelian tensor square of special orthogonal groups and spin groups are stated in the following theorem:

*Main theorem*

Let $F_q$ be a finite field with $q$ elements, $|F_q| > 4$. Then
($i$) $Spin_n(F_q) \otimes Spin_n(F_q) \cong Spin_n(F_q)$
($ii$) $SO_n(F_q) \otimes SO_n(F_q) \cong Spin_n(F_q)$.

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*Preliminaries*

This section includes some preparatory definitions and basic results on the Schur multiplier and nonabelian tensor square of groups which are used for proving our main theorem.

*Definition* 1: (Wilson, 2010) An $n \otimes n$ matrix $A$ is an orthogonal matrix if $AA^T = I$, where $A^T$ is the transpose of $A$ and $I$ is the identity matrix.

*Definition* 2**:** (Wilson, 2010) The orthogonal group of degree $n$ over a field $F_q$ consisting $q$ elements, $O_n(F_q)$, is the group of $n \otimes n$ orthogonal matrices with entries from $F_q$, with the group operation of matrix multiplication.

*Definition* 3: (Wilson, 2010) A finite group $G$ is quasisimple if $G = [G,G]$ and $G = Z(G)$ is a simple group.

*Definition* 4: (Wilson, 2010) A group $G$ is a subnormal subgroup of $H$ if there is a normal series from $G$ to $H$.

*Definition* 5: (Wilson, 2010) A group $G$ is a component of $H$ if $G$ is a quasisimple group which is a subnormal subgroup of $H$.

*Definition* 6: (Wilson, 2010) The special orthogonal group $SO_n(F_q)$ is the component of orthogonal group $O_n(F_q)$ containing the identity.

*Definition* 7: (Wilson, 2010) The spin group $Spin_n(F_q)$ is the double cover of the special orthogonal group $SO_n(F_q)$, such that there exists a short exact sequence of Lie groups

$$1 \longrightarrow C_2 \longrightarrow Spin_n(F_q) \longrightarrow SO_n(F_q) \longrightarrow 1.$$

If $R \longrightarrow F \longrightarrow G$ is a presentation of a group $G$, then $M(G) \cong (F' \cap R) = [F,R]$ (Hopf's Formula).

According to (Karpilovsky, 1987) a group $G^*$ is said to be a covering group of $G$ if $G^*$ has a subgroup $A$ such that

(i) $A \subseteq Z(G^*) \cap [G^*, G^*]$,
(ii) $A \cong M(G)$,
(iii) $G \cong G^*/A$.

A central extension of a group $G$ is a short exact sequence of groups

$$1 \longrightarrow A \xrightarrow{\alpha} H \xrightarrow{\beta} G \longrightarrow 1$$

such that α: $A \longrightarrow H$ and $\alpha(A)$ is in the $Z(H)$, the center of the group *H.*

Let $G$ be a finite fixed group and let

$$E:\ 1 \longrightarrow A \xrightarrow{\alpha} H \xrightarrow{\beta} G \longrightarrow 1$$

be a central extension. Given another central extension by $G$,

$$E_1:\ 1 \longrightarrow A \xrightarrow{\alpha'} K \xrightarrow{\beta'} G \longrightarrow 1$$

We say that $E$ covers (respectively, uniquely covers) $E_1$ if there is homomorphism (respectively, unique homomorphism) $\gamma : H \longrightarrow K$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & G & \longrightarrow & 1 \\ & & \downarrow 1_A & & \downarrow \gamma & & \downarrow 1_G & & \\ 1 & \longrightarrow & B & \xrightarrow{\alpha'} & K & \xrightarrow{\beta'} & G & \longrightarrow & 1 \end{array}$$

We shall refer to the central extension $E$ as being universal if it uniquely covers any central extension by $G$.

*Theorem* 1. (Steinberg, 1968) *If q is finite and* $|q| > 4$*, then* $M(Spin_n(F_q)) = 1$ *and the natural central extension* $Spin_n(F_q) \longrightarrow SO_n(F_q)$ *is universal.*

A group $G$ is said to be perfect if $[G,G] = G$. In the following theorem, the Schur multiplier and covering group of a finite perfect group is stated.

*Theorem* 2. (Steinberg, 1968) *Let G be a finite perfect group and*

$$1 \longrightarrow A \longrightarrow G^* \longrightarrow G \longrightarrow 1$$

*is a universal central extension, then* $A \cong M(G)$ *and* $G^*$ *is a covering group of G.*

*Proof of main theorem*

*Lemma* 1**.** *If q is finite and* $|q| > 4$*, then spin groups* $Spin_n(F_q)$ *and special orthogonal groups* $SO_n(F_q)$ *are perfect.*

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Proof : Let $\alpha : Spin_n(F_q) \to SO_n(F_q)$ be the natural universal central extension. We define $\beta : Spin_n(F_q) \times Spin_n(F_q)/(Spin_n(F_q))' \to SO_n(F_q)$ such that $\beta(A, b) = \alpha(A)$, $A \in Spin_n(F_q)$ and $b \in Spin_n(F_q) = (Spin_n(F_q))'$. We also define $\gamma_1 : Spin_n(F_q) \to Spin_n(F_q) \times Spin_n(F_q) = (Spin_n(F_q))'$ and $\gamma_2 : Spin_n(F_q) \to Spin_n(F_q) \times Spin_n(F_q) = (Spin_n(F_q))'$ such that $\gamma_1(A) = (A, 1)$ and $\gamma_2(A) = (A, A + (Spin_n(F_q))')$. Since $\beta\gamma_1 = \alpha$ and $\beta\gamma_2 = \alpha$, then $\gamma_1 = \gamma_2$. Thus $Spin_n(F_q)/(Spin_n(F_q))' = 1$. Thus $Spin_n(F_q) = (Spin_n(F_q))'$.

Since $\alpha: Spin_n(F_q) \longrightarrow SO_n(F_q)$ is an universal central extension, it is an immediate consequence that special orthogonal groups are perfect.

Proof of Main Theorem: Let $F_q$ be a finite field with $q$ elements and $|F_q| > 4$.

(*i*) According to Lemma 1, spin groups $Spin_n(F_q)$ are perfect and by Theorem 2. $M(Spin_n(F_q)) = 1$. Then we have the following short exact sequence:

$$1 \longrightarrow J_2(Spin_n(F_q)) \longrightarrow M(Spin_n(F_q)) \to 1.$$

Thus $J_2(Spin_n(F_q)) = 1$. By the following central extension,

$$1 \longrightarrow Spin_n(F_q) \longrightarrow Spin_n(F_q) \to Spin_n(F_q))' \longrightarrow 1$$

it is clear that :

$$Spin_n(F_q) \otimes Spin_n(F_q) \cong (Spin_n(F_q))' = Spin_n(F_q).$$

(*ii*) Since

$(SO_n(F_q))' = SO_n(F_q)$, $SO_n(F_q) = (SO_n(F_q))' = 1$ and so $Im\,\psi = 1$, where

$$\psi: \Gamma(SO_n(F_q))^{ab} \longrightarrow SO_n(F_q) \otimes SO_n(F_q)$$

is the homomorphism such that $\psi\gamma(A(SO_n(F_q))') = A \otimes A$ and $A \in SO_n(F_q)$. Hence, from Fig 1., $SO_n(F_q) \otimes SO_n(F_q)$ is a central extension of $SO_n(F_q)$ by $M(SO_n(F_q))$. By the relations

$$AB \otimes C = ({}^{A}B \otimes {}^{A}C)(A \otimes C) \text{ and}$$

$$A \otimes CD = (A \otimes C)({}^{C}A \otimes {}^{C}D)$$

the nonabelian tensor square $SO_n(F_q) \otimes SO_n(F_q)$ is generated by elements $A \otimes B$ with $A$ and $B$ commutators. From $[A \otimes C, B \otimes D] = (A\,{}^{C}A{-}1) \otimes ({}^{B}DD{-}1)$, it follows that an element is a commutator in $SO_n(F_q) \otimes SO_n(F_q)$. Therefore, $SO_n(F_q) \otimes SO_n(F_q)$is perfect and thus is isomorphic to its covering group, that is, $Spin_n(F_q)$.

## References


1. Brown R, Johnson DL and Robertson EF (1987) Some computations of nonabelian tensor products of groups. *J. Algebra*.11, 177-202.
2. Brown R and Loday JL (1984) Excision homotopique en basse dimension. *C. R. Acad. Sci. Paris Ser. I Math.* 298, 353-356.
3. Brown R and Loday JL (1987) Van Kampen theorems for diagrams of spaces. *Topology*. 26, 311-335.
4. Erfanian A, Rezaei A and Jafari SH (2008) Computing the nonabelian tensor square of general linear groups. *Italian J. Pure & Appl. Math.* 24, 203-210.
5. Kappe LC (1999) Non abelian tensor products of groups, the commutator connection. *Proceedings Groups St Andrews at Bath 1997*, Lecture Notes LMS. 261, 447-454.
6. Hannebauer T (1990) On nonabelian tensor square of linear groups. *Arch. Math.* 55, 30-34.
7. Karpilovsky G (1987) The schur multiplier, Clarendon Press, Oxford.
8. Rashid S, Sarmin NH, Erfanian A and Mohd Ali NM (2011a) The nonabelian tensor square ($G \otimes G$) of symplectic groups and projective symplectic groups. *Scientific Res. Essays*. 6(24), 5261-5264.
9. Rashid S, Sarmin NH, Erfanian A and Mohd Ali NM (2011b) On the nonabelian tensor square and capability of groups of order $p^2q$. *Arch. Math.* 97, 299-306.
10. Steinberg R (1968) Lectures on chevalley groups, mimeographed lecture notes. Yale University Notes, New Haven.
11. Wilson RA (2010) The finite simple groups. In: *Graduate Texts in Maths*. Springer London Dordrecht Heidelberg NY.
12. Whitehead JHC (1950) A certain exact sequence. *Ann. Math.* 52, 51-110.