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Sequence of Integers Generated by Summing the Digits of their Squares

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Abstract

Objectives: To establish some properties of sequence of numbers generated by summing the digits of their respective squares. **Methods/Analysis:** Two distinct sequences were obtained, one is obtained from summing the digits of squared integers and the other is a sequence of numbers can never be obtained be obtained when integers are squared. Also some mathematical operations were applied to obtain some subsequences. The relationship between the sequences was established by using correlation, regression and analysis of variance. **Findings:** Multiples of 3 were found to have multiples of 9 even at higher powers when they are squared and their digits are summed up. Other forms are patternless, sequences notwithstanding. The additive, divisibility, multiplicative and uniqueness properties of the two sequences yielded some unique subsequences. The closed forms and the convergence of the ratio of the sequences were obtained. Strong positive correlation exists between the two sequences as they can be used to predict each other. Analysis of variance showed that the two sequences are from the same distribution. **Conclusion/Improvement:** The sequence generated by summing the digits of squared integers can be known as Covenant numbers. More research is needed to discover more properties of the sequences.

Keywords: Digits, Factors, Multiples, Sequence of Integers, Squares, Subsequence

1. Introduction

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144...(A).

This sequence is called the square number (integers) which can be found on the online encyclopedia of integer sequence *A000290 – OEIS.*

Also the sum of square;

0, 5, 14, 30, 55, 91, 140...(B) can be found in *A000330–OEIS.*

Equation (B) is obtained from (A) and some few examples are as follows: 5 = 1 + 4, 14 = 5 + 9, 30=14+16

The square number is the square of number (in this case an integer) and is the outcome when an integer is multiplied with itself[1].

Many authors have worked on the square number but this paper introduces a new concept/property of the square number (integers) by investigating and examines the phenomenon of summing up the digits of squared numbers. Weissten[2] enumerated some characteristics of the square number while some theoretical aspects can be found in[3]. Some other literatures about the square numbers are as follows: Consecutive integers with equal sum of squares[4].

Mixed sum of Squares and Triangular Numbers[5–8].
The Sum of digits of some Sequence[9, 10].
The sum of Digits function of Squares[11,12].
Reducing a set of subtracting squares[13].
Squares of primes[14].
Sequences of squares with constant second differences[15].
Relationship between sequences and polynomials[16].
Square free numbers[17].
The sum of squares and some sequences[18].

Number sequences have been applied in real life in modeling, simulation and development of algorithms of some carefully studied phenomena[19,20].

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All these and more contributions too numerous to mention had yielded a well-documented characteristics of square numbers (integers) as follows;

- It is non-negative. $x < 0 : x^2 > 0$ and $x \geq 0 : x^2 \geq 0$.
- It increases as the integers increases.
- The ratio of two square integers is also a square. $\frac{4}{9} = \left(\frac{2}{3}\right)^2, \frac{25}{16} = \left(\frac{5}{4}\right)^2, \frac{81}{100} = \left(\frac{9}{10}\right)^2$
- A square number is also the sum of two consecutive triangular numbers[21,22].
- Square number has an odd number of positive divisors[23]. Square Divisors Number

Square	Divisors	Number	
9, 4	4, 2, 1	3	
9	9, 3, 1	3	
16	16, 8, 4, 2, 1	5	
36	36, 18, 12, 9, 6, 4, 3, 2, 1	9	etc.

- A square number cannot be a perfect number[23].

4 The proper divisors of 4 are 2 and 1

$$2 + 1 \neq 4$$

9 The proper divisors of 9 are 3, 2 and 1

$$3 + 2 + 1 \neq 9$$

16 The proper divisors of 16 are 8, 4, 2 and 1

$$8 + 4 + 2 + 1 \neq 16$$

- The only non-trivial Square Fibonacci number is $12^2 = 144$.

2. Methodology

The first 3000 integers are squared and their respective digits summed up. The first 10 **numbers**, their square and the sum of their respective digits are summarized in Table 1.

Table 1. The first ten terms, their square and digits sum

Number	Square	Sum of the Digits of the Squared Number
1	1	1
2	4	4
3	9	9
4	16	7
5	25	7
6	36	9
7	49	13
8	64	10
9	81	9
10	100	1

3. Findings

Since the first 3000 integers are used, it was observed that a sequence of numbers is obtained and can be grouped in two distinct ways. First, when an integer is squared, and the digits summed, the following numbers can be obtained at varying frequencies which form the following sequence; 1, 4, 7, 9, 10, 13, 16, 18, 19, 22, 25, 27, 28, 31, 34, 36, 37, 40, ... (C). Second, when an integer is squared and the digits summed up, the following numbers cannot be obtained which forms the second sequence; 2, 3, 5, 6, 8, 11, 12, 14, 15, 17, 20, 21, 23, 24, ... (D).

3.1 The Patternless Nature of the Sequences of Odd and Even Integers when the Digits of their Squares are Summed

Table 2 shows the results when the numbers are divided into two distinct equivalence classes of the odd and even integers.

There is no significance pattern of sequence formed by each class except the multiples of 3. "*Hence we state that an even integer when squared and its digits summed yields even or odd integer and the same applies to any odd integer*".

Table 2. The sum of the digits for odd and even numbers

Odd Number	Sum of the Digits of the Squared Number	Even Number	Sum of the Digits of the Squared Number
401	16	402	18
403	22	404	19
405	18	406	28
407	31	408	27
409	25	410	16
411	27	412	31
413	28	414	27
415	19	416	22
417	36	418	25

3.1.1 Multiples of 3

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36 ... (E).

Table 3 shows some integers multiples of 3, their square and their respective sum of digits:

"Hence we state that any integer divisible by 3, if squared and its digits summed yields an integer divisible by 9".

3.1.2 Higher Powers of Multiples of 3

Even at higher powers of the multiples of 3, the same result is obtained as shown in Table 4.

3.2 Characteristics of the Two Sequences (C) and (D)

- (C) ∪ (D) ∪ 0 = □
- From Fibonacci sequence; *A000045 – OEIS.* 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 ... (F)

  Sequence (C) contains 1, 1, 13, 34, 55, ... (FA)
  Sequence (D) contains 2, 3, 5, 8, 21, 89 ... (FB)

- From Lucas sequence; *A000032 – OEIS 2, 1, 3, 4, 7, 11, 18,* 29, 47, 76, 123, 199, 322, ... (G)

Table 3. Some integers multiples of 3

Number	Square	Sum of the Digits of Square
3	9	9
21	441	9
24	576	18
57	3249	18
63	3969	27
447	199809	36

Table 4. Higher powers of multiple of 3 and their sums of digits.

X	x^3	sum of digits	x^4	sum of digits	x^5	sum of digits
3	27	9	81	9	243	9
6	216	9	1296	18	7776	27
9	729	18	6561	18	59049	27
12	1728	18	20736	18	248832	27
15	3375	18	50625	18	759375	36
18	5832	18	104976	27	1889568	45

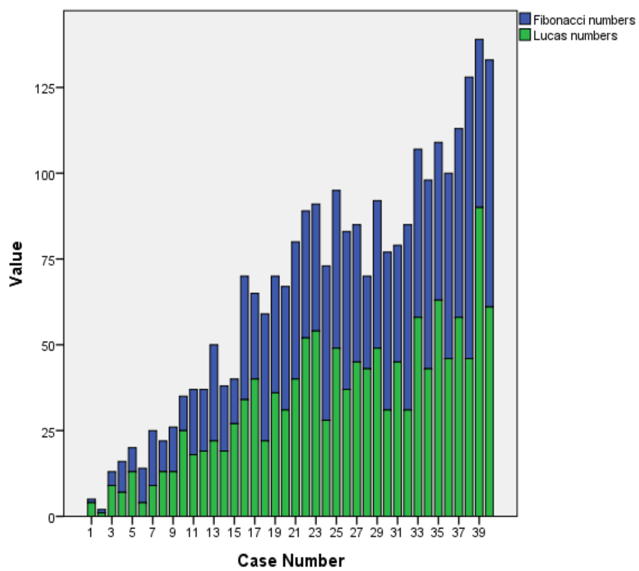

Figure 1. Component bar chart of the first 40 Fibonacci and Lucas number.

1 1 10 100
4 2 11 20
7 4 5 32 40 49 50
9 3 6 9 12 15 18 21 30 39 45 48 51 60 90
10 8 19 35 46 55 71 80
13 7 16 25 29 34 38 47 52 56 61 65 70 79
16 13 14 22 23 31 41 58 59 68 85 95
18 24 27 33 36 42 54 57 66 69 72 75 78 81 84 96 99
19 17 26 28 37 44 53 62 64 73 82 89 91 98
22 43 74 88 92 97
25 67 76 77 86 94
27 63 87 93
28
31 83
34
36

Figure 2. The first 100 numbers and their digits sum grouped in sequence C.

Sequence (C) contains 1, 4, 7, 18, 76, ... (GA)
Sequence (D) contains 2, 3, 11, 29, 47, ... (GB)

- The first 40 numbers of both Fibonacci and Lucas sequences were squared, the sum of their respective individual numbers were obtained and the results are represented in a component bar chart.

As seen from the chart, the Lucas numbers increases more rapidly than the Fibonacci numbers.

3.2.1 Subsequences of Sequence C

Each of the numbers of sequence C also forms a sequence. For example, the first 100 natural numbers can be grouped based on the numbers in sequence C.

3.2.2 Additive Properties

- Addition of two numbers of sequence (C) can yield numbers in both sequences (C) and (D).
- Addition of two numbers of sequence (C) can produce numbers in the same sequence if;
  (a) A multiple of 9 is added to any numbers ofsequence (C).
  (b) A multiple of 9 are added to each other.

- A pattern can be formed from the addition of the numbers of sequence which can be seen from Table 1.
- Addition of two numbers of sequence (D) yield no pattern but a patterned triangle similar to Paschal can be obtained which contained some numbers of sequence (C) in unique arrangement.

3.2.3 Multiplicative Properties

- The multiplication of any two numbers of sequence (C) yield a number in the same sequence.

Table 5. Addition of terms of sequence C.

Addition	1	4	7	9	10	13	16	18
1	2	5	8	10	11	14	17	19
4	5	8	11	13	14	17	20	22
7	8	11	14	16	17	20	23	25
9	10	13	16	18	19	22	25	27
10	11	14	17	19	20	23	26	28
13	14	17	20	22	23	26	29	31
16	17	20	23	25	26	29	32	34
18	19	22	25	27	28	31	34	36

4
5 5
7 6 **7**
8 8 8 8
10 **9** **10** **9** **10**
13 11 11 11 11 **13**
14 14 **13** 12 **13** 14 14

Figure 3. Binomial table obtained from addition of terms in sequence D.

4
6 6
10 **9** **10**
12 15 15 12
16 **18** **25** **18** **16**
22 24 30 30 24 **22**
24 33 **40** **36** **40** 33 24

Figure 4. Binomial table formed from multiplication of terms in sequence D.

- The multiplication of any two numbers of sequence (D) does not necessarily yield a number in the sequence.
- 3. Multiplication of two numbers of sequence (D) yield no pattern but a patterned triangle similar to Paschal can be obtained which contained some numbers of sequence (C) in unique arrangement.

3.2.4 Divisibility Properties

- Every 4th number of the sequence (C) is a multiple of 9.
- As expected all the square numbers are in sequence (C).
- All three consecutive numbers of sequence (C) are coprime but not pairwise $\gcd(a,b,c)=1 \;\; a,b,c \in C$. $\gcd(a,b)=\gcd(a,c)=\gcd(b,c)\neq 1$.
- All four consecutive numbers of sequence (C) are coprime but not pairwise $\gcd(a,b,c,d)=1 \;\; a,b,c,d \in C$. $\gcd(a,b)=\gcd(a,c)=\gcd(a,d)=\gcd(b,c)=\gcd(b,d)=\gcd(c,d)\neq 1$.

3.3 Uniqueness of Sequences C and D

Sequences (C) and (D) are unique. The complete respective sequences cannot be obtained by increment or decrement of the numbers in the sequences rather various sequences is obtained. When 1 is added to all the numbers in sequence (C), we obtain; 2, 5, 8, 10, 11, 14, 17, 19, 20, 23, 26, 28, 29, 32, 35, 37, 38, 41, 44... (H). When 2 is added to all numbers in sequence (C), we obtain; 3, 6, 9, 11, 12, 15, 18, 20, 21, 24, 27, 29, 30, 33, 36, 38, 39, 42, 45 ... (HA). When 3 is added is added to all numbers in sequence (C), we obtain; 4, 7, 10, 12, 13, 16, 19, 21, 22, 25, 28, 30, 31, 34, 37, 39, 40, 43, 46, ... (HB). Here it can be seen that sequence (HB) is closely related to sequence (C). When 1 is subtracted from all numbers in sequence (C), we obtain; 0, 3, 6, 8, 9, 12, 15, 17, 18, 21, 24, 26, 27, 30, 33, 35, 36, 39, 42, ... (HC). When 2 is subtracted from all numbers in sequence (C), we obtain; −1, 2, 5, 7, 8, 11, 14, 16, 17, 20, 23, 25, 26, 29, 32, 35, 38, 41, ... (HD). When 3 is subtracted from all numbers in sequence (C), we obtain; −2, 1, 4, 6, 7, 10, 13, 15, 16, 19, 22, 24, 25, 28, 31, 33, 34, 37, 40, ... (HE) Here it can be seen that sequence (HE) is closely related to sequence (C). When 1 is added to all the numbers in sequence (D), we obtain; 3, 4, 6, 7, 9, 12, 13, 15, 16, 18, 21, 22, 24, 25, 27, 30, 31, 33, 34, ... (I). When 2 is added to all the numbers in sequence (D), we obtain; 4, 5, 7, 8, 10, 13, 14, 16, 17, 19, 22, 23, 25, 26, 28, 31, 32, 34, 35, ... (IA). When 3 is added to all the numbers in sequence (D), we obtain; 5, 6, 8, 9, 11, 14, 15, 17, 18, 20, 23, 24, 26, 27, 29, 32, 33, 35, 36, ... (IB). Here it can be seen that sequence (IB) is closely related to sequence (D). When 1 is subtracted from all the numbers in sequence (D), we obtain; 1, 2, 4, 5, 7, 10, 11, 13, 14, 16, 19, 20, 22, 23, 25, 28, 29, 31, 32, ... (IC). When 2 is subtracted from all the numbers in sequence (D), we obtain; 0, 1, 3, 4, 6, 9, 10, 12, 13, 15, 18, 19, 21, 22, 24, 27, 28, 30, 31, ... (ID). When 3 are subtracted from all the numbers in sequence (D), we obtain; −1, 0, 2, 3, 5, 8, 9, 11, 12, 14, 17, 18, 20, 21, 23, 26, 27, 29, 30, ... (IE). Here it can be seen that sequence (IE) is closely related to sequence (D).

3.4 The Ratio of Sequences (C) and (D)

3.4.1 The Ratio of Sequence (C)

The ratio of the two successive integers of sequence (C) is as follows: $\frac{4}{1}, \frac{7}{4}, \frac{9}{7}, \frac{10}{9}, \frac{13}{10}, \frac{16}{13}, \frac{18}{16}, \ldots$ (J)

The sequence converges to almost one with a mean of 1.11200275. The closed form solution of the ratio can be written as: $\varphi = \frac{687}{250}\left[\frac{1+\sqrt{5}}{8}\right] \approx 1.112$

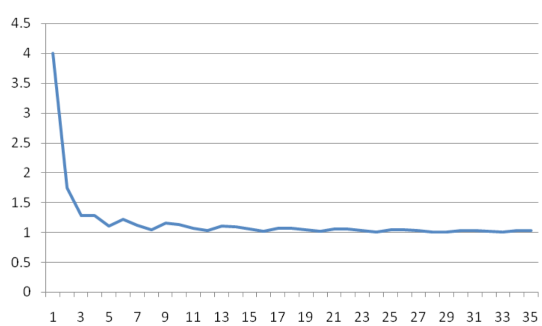

Figure 5. The ratio of sequence (C). x axis - terms in sequence C; y axis - the ratio of 2 consecutive terms of sequence C.

3.4.2 The Ratio of Sequence (D)

The ratio of the two successive integers of sequence (C) is as follows: $\frac{3}{2}, \frac{5}{3}, \frac{6}{5}, \frac{8}{6}, \frac{11}{8}, \frac{12}{11}, \frac{14}{12}, \ldots$ (K)

The sequence converges to almost one with a mean of 1.101494025. The closed form solution of the ratio can be written as: $\varphi = \frac{681}{250}\left[\frac{1+\sqrt{5}}{8}\right] \approx 1.1019.$

3.5 The Sequences Obtained from the Various Factors of Sequences (C) and (D)

The first 40 members of sequences C and D are listed. Some subsequences are obtained by the various factors such as 2n, 3n, 4n...

3.5.1 Factors of 2

Subsequence is formed for both sequences C and D if they are arranged based on the factors of two. The first is

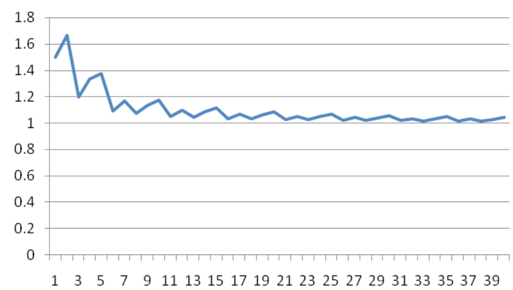

Figure 6. The ratio of sequence (D). x axis - terms in sequence D; y axis - the ratio of 2 consecutive terms of sequence D.

Table 6. The first 10 terms of sequences C and D

n	1	2	3	4	5	6	7	8	9	10
C	1	4	7	9	10	13	16	18	19	22
D	2	3	5	6	8	11	12	14	15	17

Table 7. The 11th to 20th terms of sequences C and D

n	11	12	13	14	15	16	17	18	19	20
C	25	27	28	31	34	36	37	40	43	45
D	20	21	23	24	26	29	30	32	33	35

Table 8. The 21st to 30th terms of sequences C and D

n	21	22	23	24	25	26	27	28	29	30
C	46	49	52	54	55	58	61	63	64	67
D	38	39	41	42	44	47	48	50	51	53

Table 9. The 31st to 40th terms of sequences C and D

n	31	32	33	34	35	36	37	38	39	40
C	70	72	73	76	79	81	82	85	88	90
D	56	57	59	60	62	65	66	68	69	71

for sequence C and the second is for sequence D. 4, 9, 13, 18, 22, 27, 31, 36, 40, 45, ... (L) 3, 6, 11, 14, 17, 21, 24, 29, 32, 35, ... (M).

3.5.2 *Factors of 3*

7 ,13, 19, 27, 34, 40, 46, 54, 61, 67, ... (N) 5, 11, 15, 21, 26, 32, 38, 42, 48, 53, ... (O).

3.5.3 *Factors of 4*

9, 18, 27, 36, 45, 54, 63, 72, 81, 90, ...(P) 6, 14, 21, 29, 35, 42, 50, 57, 65, 71, ... (Q).

3.6 The Square of Sequences C and D

New sequences are obtained from the square of sequences C and D.

3.6.1 *The Square of Sequence C*

1, 16, 49, 81, 100, 169, 324, 361, 484, ...(R).

3.6.2 *The Square of Sequence D*

4, 9, 25, 36, 64, 121, 144, 196, 225, 289, ...(S).

3.7 The Ratio of the Sequences

The ratio of the two sequences also produced some sequences.

3.7.1 *The Sequence C/D*

$$\frac{1}{2},\frac{4}{3},\frac{7}{5},\frac{9}{6},\frac{10}{8},\frac{13}{11},\frac{16}{12},\ldots \text{(T)}.$$

3.7.2 *The Sequence D/C*

$$\frac{2}{1},\frac{3}{4},\frac{5}{7},\frac{6}{9},\frac{8}{10},\frac{11}{13},\frac{12}{16},\ldots \text{(U)}$$

3.8 Linear Correlation between Sequences C and D

There is a strong positive correlation between the two sequences. Pearson correlation coefficient is 0.999, Spearman rho is 1.0 and Kendall's tau is 1.0.

3.9 Regression Analysis of the First 40 Terms of Sequences C and D

Since there is a strong positive correlation between the two sequences, the predictive capability of the sequences with respect to each other is analyzed using the regression for the first 40 terms of both sequences.

3.9.1 *Sequence C as the Dependent Variable*

The results of regression analysis of the two sequences when sequence C is the dependent variable and sequence D as the independent variable are summarized as follows;

The R, adjusted R square, R square and R square change have the same value of 0.999. The regression equation is; $C = 0.201 + 0.999D$ (1). The result of the analysis of variance is summarized in Table 10.

3.9.2 *Sequence D as the Dependent Variable*

The results of regression analysis of the two sequences when sequence D is the dependent variable and sequence C as the independent variable are summarized as follows; The R, adjusted R square, R square and R square change have the same value of 0.999. The regression equation is; $D = -0.123 + 0.004C$ (2). The result of the analysis of variance is summarized in Table 11.

3.10 Test of Equality of Means

The sequences have the same mean effect as summarized in Table 12.

Table 10. ANOVA Table 1

ANOVA[a,c]						
Model		Sum of Squares	Df	Mean Square	F	Sig.
1	Regression	27069.771	1	27069.771	37096.437	.000[b]
	Residual	27.729	38	.730		
	Total	27097.500	39			

a. Dependent Variable: C b Predictors; (Constant), D.

Table 11. ANOVA Table 2

ANOVA[a]						
Model		Sum of Squares	Df	Mean Square	F	Sig.
1	Regression	17174.807	1	17174.807	37096.437	.000[b]
	Residual	17.593	38	.463		
	Total	17192.400	39			

a. Dependent Variable: D b. Predictors: (Constant), C.

Table 12. ANOVA table for the test of equality of means of C and D.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	$F_{calculated}$	$F_{tabulated}$
Between Groups	1786.05	1	1786.05	3.145455	3.963472
Within Groups	44289.9	78	567.8192		
	46075.95	79			

4. Conclusion

The paper have described the properties of sum of the digits of square numbers and their associated sequences and multiples of 3 were found to be the only class of integers with unique pattern when their digits of their square are summed. The closed form of the ratios gave approximate ratios. More research is needed to produce more features and properties of the sequences. The authors proposed that sequence C be named COVENANT NUMBERS and be included in the online encyclopedia of integer sequences database.

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