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# Convergence Theorems for Nonspreading Type Mappings in Hilbert Spaces

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## Abstract

In this paper first we give a weak convergence ergodic theorem for K-strictly pseudo-nonspreading and nonspreading mapping in Ishikawa iteration scheme, motivated by Kurokawa and Takahashi in[9]. Then we deal with a strong convergence theorem for nonspreading mappings in a Hilbert space. Our results improve and extend Kurokawa and Takahashi result.

**Keywords:** Fixed Point, k-Strictly Peseudo-Nonspreading Mapping, Nonspreading Mapping, Strong Convergence, Weak Convergence

## 1. Introduction

Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. A mapping $T: C \to C$ is said to be nonexpansive if $\| Tx - Ty \| \le \| x - y \|$ for all $x, y \in C$. The set of fixed points of $T$ is denoted by $F(T)$. A mapping $T: C \to C$ with $F(T) \neq \phi$ is called quasi-nonexpansive if

$$\| x - Ty \| \le \| x - y \|$$

for all $x \in F(T)$ and $y \in C$. It is well known that the set of fixed points of a quasi-nonexpansive mapping $T$ is closed and convex, see[6]. Recently, Kohosaka and Takahashi[8] introduced the following nonlinear mapping: Let $E$ be a smooth, strictly convex and reflexive Banach space, let $J$ be the duality mapping of $E$ and let $C$ be a nonempty closed convex subset of $E$. Then, a mapping $T: C \to C$ is said to be nonspreading if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where $\phi(y, x) = \| x \|^2 - 2(x, Jy) + \| y \|^2$ for all $x, y \in E$. They considered such mapping to study the resolvents of a maximal monotone operator in the Banach space. In the case when $E$ is a Hilbert space, we know that $\phi(x, y) = \| x - y \|^2$ for all $x, y \in E$. So, a nonspreading mapping $T: C \to C$ in a Hilbert space $H$ is defined as follows:

$$2 \| Tx - Ty \|^2 \le \| Tx - y \|^2 + \| Ty - x \|^2 \tag{1}$$

It is shown in[5] that (1) is equivalent to

$$\| Tx - Ty \|^2 \le \| x - y \|^2 + 2 < x - Tx, y - Ty>, \tag{2}$$

for al1l $x, y \in C$. Observe that if $T$ is nonspreading and $F(T) \neq$ f, then $T$ is quasi-nonexpansive.

A mapping $T: C \to C$ is said to be k-strictly pseudo-nonspreading if there exsists $k \in [0, 1]$ such that

$$\| Tx - Ty \|^2 \le \| x - y \|^2 + 2 < x - Tx, y - Ty> + k \| x - (y - Ty) \|^2 \tag{3}$$

for all $x, y \in C$. Clearly, every nonspreading mapping is k-strictly pseudo-nonspreading.

The following example shows that the class of k-strictly pseudo-nonspreading mapping is more general than the class of nonspreading mapping.

### 1.1 Example

(See[11]). Let $\mathbb{R}$ denote the real numbers with the usual norm. Let $T: \mathbb{R} \to \mathbb{R}$ be defined for each $x \in \mathbb{R}$ by

$$Tx = \begin{cases} x & if\ x \in (-\infty, 0), \\ -2x & if\ x \in [0, \infty). \end{cases}$$

Then, $T$ is k-strictly pseudo-nonspreading but not nonspreading mapping.

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## 1.2 Lemma

(See[11]). Let $C$ be a nonempty closed convex subset of real Hilbert space $H$, and $T: C \to C$ be a k-strictly pseudo-nonspreading mapping. If $F(S) \neq \phi$, then it is closed and convex.

Some iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first iteration processes is known as Halpern iteration process[3]. The following strong convergence theorem of Halpern's type was proved by Wittmann: for any $x_1 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) Tx_n \qquad n \in \mathbb{N} \cup \{0\}$$

for all $n \in \mathbb{N}$, where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in [0,1], $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n x = \infty$ and $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.

The second iteration process is introduced by Mann[10]. We also know the following weak convergence theorem of Mann's type: for any $x_1 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Tx_n \qquad n \in \mathbb{N} \cup \{0\}$$

for all $n \in \mathbb{N}$, where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in [0,1] and $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then $\{x_n\}$ converges weakly to a fixed point of $T$.

The third iteration process[7] is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1+\alpha_n)Ty_n, \end{cases}$$

where the initial guess $x_0$ is taken in $C$ arbitrarily and the sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are in [0,1].

We know the following first nonlinear ergodic theorem in a Hilbert space from Baillon[2].

## 1.3 Theorem

Let $C$ be a nonempty bounded closed convex subset of $H$ and let $T: C \to C$ be nonexpansive. Then for any $x \in C$, $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converges weakly to an element $z \in F(T)$.

Kurokawa and Takahashi in proved the following weak convergence nonlinear ergodic theorem of Baillon's type for nonspreading mappings in a Hilbert space. They used Halpern's iteration scheme.

## 1.4 Theorem

(See [9]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a nonspreading mapping from $C$ into itself. Define two sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ z_n = \dfrac{1}{n}\sum_{k=1}^{n} x_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\alpha_n \to 0$ and $0 \leq \alpha_n \leq 1$. If $F(T) \neq \phi$, then $\{z_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$ and $P$ is the metric projection of $H$ onto $F(T)$.

In particular, for any $x \in C$, define

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} \text{ and } T^k x$$

Then $\{S_n x\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} PT^n x$.

The following theorem, is the strong convergence theorem for nonspeading mappings in a Hilbert space that Kurokawa and Takahashi proved.

## 1.5 Theorem

(See[9]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a nonspreading mapping from $C$ into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + \left(1-\alpha_n\right) z_n, \\ z_n = \dfrac{1}{n}\sum_{k=0}^{n-1} T^k x_n, \end{cases}$$

for all $n \in \mathbb{N}$, where, $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T) \neq \phi$, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu$, where $P$ is the metric projection of $H$ onto $F(T)$.

Recently Kurokawa and Takahashi in[9] obtained a weak mean convergence ergodic theorem of Baillon's type for nonspreading mappings in a Hilbert space. They used Halpern's iteration scheme for proving their theorem. In this paper motivated by them, first we give a nonlinear ergodic theorem for k-strictly peseudo-nonspreading and nonspreading mapping in Ishikawa iteration scheme. Then we deal with a strong convergence theorem for nonspreading mappings in a Hilbert space. These two theorems improve and generalize Kurokawa-Takahashi theorems.

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively. In a Hilbert space $H$, it is known that

i) $\| tx + (1-t)\, y \|^2 = t \| x \|^2 + (1-t) \| y \|^2 - t(1-t) \| x-y \|^2$ (4)
for all $x, y \in H$ and for all $t \in [0,1]$ see[4].

ii) $\| x+y \|^2 \le \| x \|^2 + 2\,(y, x+y) \quad \forall x, y \in H,$ (5)

iii) $\| y \|^2 - \| x \|^2 \le 2\,(y-x, y) \quad \forall x, y \in H.$

See[14].

Let $\{x_n\}$ be a sequence in $H$ and $x \in H$. Weak convergence of $\{x_n\}$ is denoted by $x_n \underline{w}\, x$ and strong convergence by $x_n \to x$. Let $H$ be a nonempty closed convex subset of a real Hilbert space $H$. The nearest point projection of $H$ onto $C$ is denoted by $P_c$, that is, for all $x \in H$ and $Y \in C$

$$\| x - P_c x \| \le \| x - y \|.$$

Such $P_c$ is called the metric projection of $H$ onto $C$. It is known that for each $x \in H$

$$\left\langle x - P_C x,\, y - P_C x \right\rangle \le 0 \qquad \forall y \in C. \tag{6}$$

See[12] for more details.

### 2.1 Lemma

See[1]. Let $\{S_n\}$ be a sequence of non-negative real numbers satisfying the condition

$$S_{n+1} \le (1-\alpha_n)\, S_n + \alpha_n \gamma_n + \theta_n,$$

where $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\gamma_n\}$ are real sequences such that:

i) $\alpha_n \subset [0,1]$ and $\sum_{k=0}^{\infty} \alpha_n = \infty,$

ii) $\limsup \gamma_n \le 0,$

iii) $\theta_n \ge 0$ for all $n \in \mathbb{N}$ and $\sum_{k=0}^{\infty} \theta_n < \infty.$

Then, $\lim_{n\to\infty} s_n = 0.$

### 2.2 Lemma[13]

See[13]. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $P$ be the metric projection of $H$ onto $C$ and let $\{x_n\}$ be a sequence in $H$. If $\| x_{n+1} - q \| \le \| x_n - q \|$, for all $q \in C$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.

## 3. Main Results

### 3.1 Theorem

Let $H$ be a Hilbert space, and let $C$ be a nonempty closed and convex subset of $H$. Let $T: C \to C$ be a nonspreading mapping with $F(T) \ne \phi$ and $S: C \to C$ be a k-strictly peseudo-nonspreading mapping with $F(S) \ne \phi$, such that $F(T) \cap F(S) \ne \phi$. Suppose that $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are sequences generated by $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n) T y_n, \\ y_n = \beta_n x_n + (1-\beta_n) S x_n, \\ u_n = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of (0,1) and $0 < a < \alpha_n < b < 1$, $0 \le k < c < \beta_n < d < 1$ for some $a, b, c, d > 0$. Then $\{u_n\}$ converges weakly to $u \in F(T) \cap F(S)$, where $u = \lim_{n\to\infty} Px_n$ and $P$ is the metric projection of $H$ onto $F(T) \cap F(S)$.

#### 3.1.1 Proof

We divide the proof into 6 steps.

##### 3.1.1.1 Step 1

We first show that if $z \in F(T) \cap F(S)$, then $\lim_{n\to\infty} \| x_n - z \|$ exists and $\{x_n\}$ is bounded. Since $F(T) \cap F(S) \ne \phi$ and $T$ is nonspreading, $T$ is quasi nonexpansive, then we have

$$\begin{aligned} \| x_{n+1} - z \| &= \| \alpha_n x_n + (1-\alpha_n)\, Ty_n - z \| \\ &\le \alpha_n \| x_n - z \| + (1-\alpha_n) \| Ty_n - z \| \\ &\le \alpha_n \| x_n - z \| + |\ (1-\alpha_n) \| y_n\ z \| \end{aligned} \tag{7}$$

Putting $V_n = \beta_n I + (1-\beta_n) S$, from (4) and $S$ as a k-strictly peseudo-nonspreading mapping. For all $x, y \in C$ we have

$$\begin{aligned} \| V_n x - V_n y \|^2 &= \| \beta_n (x-y) + (1-\beta_n)(Sx - Sy) \|^2 \\ &= \beta_n \| x-y \|^2 + (1-\beta_n) \| (Sx - sy) \|^2 \\ &\quad - \beta_n (1-\beta_n) \| x - Sx - (y - Sy) \|^2 \\ &\le \beta_n \| x-y \|^2 + (1-\beta_n)(\| x-y \|^2 + 2\langle x - Sx, y - Sy \rangle \\ &\quad + k \| x - Sx - (y - Sy) \|^2) \\ &\quad - \beta_n (1-\beta_n) \| x - Sx - (y - Sy) \|^2 \\ &= \| x-y \|^2 + 2(1-\beta_n) \langle x - Sx, y - Sy \rangle - (1-\beta_n)(\beta_n - k) \\ \| x - Sx - (y - Sy) \|^2 &\le \| x-y \|^2 + 2(1-\beta_n) \langle x - Sx, y - Sy \rangle \\ &= \| x-y \|^2 + \frac{2}{1-\beta_n} x - V_n x, y - V_n y. \end{aligned} \tag{8}$$

Since $V_n z = \beta_n z + (1-\beta_n)\, Sz$, then from (8) we have

$$\| Y_n - z \| = \| V_n x - z \| = \| V_n x - V_n z \| \le \| x_n - z \|, \tag{9}$$

therefore, by (7) and (9)

$$\| x_{n+1} - z \| \leq \alpha_n \| x_n - z \| + (1 - \alpha_n) \| x_n - z \| = \| x_n - z \|.$$

So, it yeilds that $\lim_{n\to\infty} \| x_n - z \|$ exist and hence $\{x_n\}$, $\{Sx_n\}$ $\{y_n\}$ and $\{Ty_n\}$ is bounded.

### 3.1.1.2 Step 2

We claim that

$$\frac{1}{n}\| x_{n+1} - Ty \|^2 \leq \| x_n - Ty \|^2 + \| Ty - y \|^2 + 2\langle u_n - Ty, Ty - y\rangle$$
$$+\frac{1}{n}\sum_{k=1}^{n} \| Sx_k - x_k \|^2 + \frac{2}{n}\sum_{k=1}^{n} \| Sx_k - x_k \| \| x_k - y \|$$
$$+\frac{2}{n}\sum_{k=1}^{n} \| Sx_k - x_k \| \| y - Ty \|$$
$$+\frac{2}{n}\sum_{k=1}^{n} \| x_k - Ty_k \| \| y - Ty \|.$$

From (4), (2) and since $0 < \alpha_n < 1$ we have for all $y \in C$ and $k \in \mathbb{N}$,

$$\begin{aligned}\| x_{k+1} - Ty \|^2 &= \| \alpha_k x_k + (1-\alpha_k)Ty_k - Ty \|^2\\ &= \| \alpha_k (x_k - Ty) + (1-\alpha_k)(Ty_k - Ty) \|^2\\ &= \alpha_k \| x_k - Ty \|^2 + (1-\alpha_k) \| Ty_k - Ty \|^2 - \alpha_k(1-\alpha_k)\\ &\quad \| x_k - Ty_k \|^2 \leq \alpha_k \| x_k - Ty \|^2 + (1-\alpha_k) \| Ty_k - Ty \|^2\\ &\leq \alpha_k \| x_k - Ty \|^2 + (1-\alpha_k)[\| y_k - y \|^2 + 2\langle y_k - Ty_k, y - Ty\rangle]\end{aligned}$$

Since

$$\begin{aligned}\| y_k - y \|^2 &= \| \beta_k x_k + (1-\beta_k)Sx_k - y \|^2\\ &\leq \beta_k \| x_k - y \|^2 + (1-\beta_k) \| Sx_k - y \|^2,\end{aligned}$$

so, we have

$$\begin{aligned}\| x_{k+1} - Ty \|^2 &\leq \alpha_k \| x_k - Ty \|^2 + (1-\alpha_k)\beta_k \| x_k - y \|^2\\ &\quad + (1-\alpha_k)(1-\beta_k) \| Sx_k - y \|^2\\ &\quad + 2(1-\alpha_k)\langle y_k - Ty_k, y - Ty\rangle\\ &= \alpha_k \| x_k - Ty \|^2 + (1-\alpha_k)\beta_k \| x_k - y \|^2\\ &\quad + (1-\alpha_k)(1-\beta_k) \| Sx_k - x_k + x_k - y \|^2\\ &\quad + 2(1-\alpha_k)\langle y_k - Ty_k, y - Ty\rangle\\ &\leq \alpha_k \| x_k - Ty \|^2 + (1-\alpha_k)\beta_k \| x_k - y \|^2\\ &\quad + (1-\alpha_k)(1-\beta_k) \| Sx_k - x_k \|^2\\ &\quad + (1-\alpha_k)(1-\beta_k) \| x_k - y \|^2\\ &\quad + 2(1-\alpha_k)(1-\beta_k)\langle Sx_k - x_k, x_k - y\rangle\\ &\quad + 2(1-\alpha_k)\langle y_k - Ty_k, y - Ty\rangle\\ &\leq \alpha_k \| x_k - Ty \|^2 + (1-\alpha_k) \| x_k - y \|^2\\ &\quad + (1-\alpha_k)(1-\beta_k) \| Sx_k - x_k \|^2\\ &\quad + 2(1-\alpha_k)(1-\beta_k)\langle Sx_k - x_k, x_k - y\rangle\\ &\quad + 2(1-\alpha_k)\langle y_k - Ty_k, y - Ty\rangle. \qquad (10)\end{aligned}$$

But, we have

$$\begin{aligned}\| x_k - y \|^2 &= \| x_k - Ty + Ty - y \|^2\\ &= \| x_k - Ty \|^2 + \| Ty - y \|^2 + 2\langle x_k - Ty, Ty - y\rangle. \qquad (11)\end{aligned}$$

So, from (10) and (11)

$$\begin{aligned}\| x_{k+1} - Ty \|^2 &\leq \alpha_k \| x_k - Ty \|^2\\ &\quad + (1-\alpha_k) \| x_k - Ty \|^2\\ &\quad + (1-\alpha_k) \| Ty - y \|^2\\ &\quad + 2(1-\alpha_k)\langle x_k - Ty, Ty - y\rangle\\ &\quad + (1-\alpha_k)(1-\beta_k) \| Sx_k - x_k \|^2\\ &\quad + 2(1-\alpha_k)(1-\beta_k)\langle Sx_k - x_k, x_k - y\rangle\\ &\quad + 2(1-\alpha_k)\langle y_k - Ty_k, y - Ty\rangle. \qquad (12)\end{aligned}$$

But we have

$$\begin{aligned}\langle y_k - Ty_k, y - Ty\rangle &= \langle y_k - x_k, y - Ty\rangle\\ &\quad + \langle x_k - Ty_k, y - Ty\rangle\\ &= \langle \beta_k x_k + (1-\beta_k)Sx_k - x_k, y - Ty\rangle\\ &\quad + \langle x_k - Ty_k, y - Ty\rangle\\ &= \beta_k \langle x_k - x_k, y - Ty\rangle\\ &\quad + (1-\beta_k)\langle Sx_k - x_k, y - Ty\rangle\\ &\quad + \langle x_k - Ty_k, y - Ty\rangle\end{aligned}$$

Thus, by (12)

$$\begin{aligned}\| x_{k+1} - Ty \|^2 &\leq \alpha_k \| x_k - Ty \|^2\\ &\quad + \left(1-\alpha_k\right) \| x_k - Ty \|^2 + \left(1-\alpha_k\right) \| Ty - y \|^2\\ &\quad + 2\left(1-\alpha_k\right)\langle x_k - Ty, Ty - y\rangle\\ &\quad + \left(1-\alpha_k\right)\left(1-\beta_k\right) \| Sx_k - x_k \|^2\\ &\quad + 2\left(1-\alpha_k\right)\left(1-\beta_k\right)\langle Sx_k - x_k, x_k - y\rangle\\ &\quad + 2\left(1-\alpha_k\right)\left[\left(1-\beta_k\right)\langle Sx_k - x_k, y - Ty\rangle + \langle x_k - Ty_k, y - Ty\rangle\right].\end{aligned}$$

Therefore

$$\begin{aligned}\| x_{k+1} - Ty \|^2 &\leq \| x_k - Ty \|^2 + \left(1-\alpha_k\right) \| Ty - y \|^2\\ &\quad + 2\left(1-\alpha_k\right)\langle x_k - Ty, y - Ty\rangle\\ &\quad + \| Sx_k - x_k \|^2 + 2\langle Sx_k - x_k, x_k - y\rangle\\ &\quad + 2\langle Sx_k - x_k, y - Ty\rangle\\ &\quad + 2\langle x_k - Ty, Ty - y\rangle.\end{aligned}$$

Summing these inequalities from $k = 1$ to $n$ and dividing by $n$, Since $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1), we have

$$\frac{1}{n}\sum_{k=1}^{n}\| x_{k+1}-Ty\|^2\le\frac{1}{n}\sum_{k=1}^{n}\| x_k-Ty\|^2+\frac{1}{n}\sum_{k=1}^{n}\| Ty-y\|^2$$
$$+\frac{2}{n}\sum_{k=1}^{n}\langle x_k-Ty,Ty-y\rangle+\frac{1}{n}\sum_{k=1}^{n}\| Sx_k-x_k\|^2$$
$$+\frac{2}{n}\sum_{k=1}^{n}\langle Sx_k-x_k,x_k-y\rangle+\frac{2}{n}\sum_{k=1}^{n}\langle Sx_k-x_k,y-Ty\rangle$$
$$+\frac{2}{n}\sum_{k=1}^{n}\langle x_k-Ty_k,y-Ty\rangle.$$

So, we have

$$\frac{1}{n}\| x_{n+1}-Ty\|^2\le\| x_n-Ty\|^2+\| Ty-y\|^2$$
$$+2\langle u_n-Ty,Ty-y\rangle+\frac{1}{n}\sum_{k=1}^{n}\| Sx_k-x_k\|^2$$
$$+\frac{2}{n}\sum_{k=1}^{n}\| Sx_k-x_k\|\,\| x_k-y\|$$
$$+\frac{2}{n}\sum_{k=1}^{n}\| Sx_k-x_k\|\,\| y-Ty\|$$
$$+\frac{2}{n}\sum_{k=1}^{n}\| x_k-Ty_k\|\,\| y-Ty\|. \tag{13}$$

### 3.1.1.3 Step 3

We prove that

$$\frac{1}{n}\| x_{n+1}-Sy\|^2\le\frac{1}{n}\| x_1-Sy\|^2+\frac{1}{n}\sum_{k=1}^{n}\| x_k-Sx_k\|^2$$
$$+\frac{2}{n}\sum_{k=1}^{n}\| Ty_k-x_k\|\,\| x_k-Sx_k\|$$
$$+\frac{(1+k)}{n}\sum_{k=1}^{n}\| Sy-y\|^2$$
$$+2\langle u_n-Sy,Sy-y\rangle$$
$$+\frac{2}{n}\sum_{k=1}^{n}\| x_k-Sx_k\|\,\| y-Sy\|$$
$$+\frac{1}{n}\sum_{k=1}^{n}\| x_k-Sx_k\|^2$$
$$+\frac{2}{n}\sum_{k=1}^{n}\| x_k-Sx_k\|\,\| Sy-y\|$$
$$+\frac{2}{n}\sum_{k=1}^{n}\| Ty_k-x_k\|\,\| Sx_k-Sy\|$$
$$+\frac{2}{n}\sum_{k=1}^{n}\| x_k-Sx_k\|\,\| Sx_k-Sy\|.$$

We have for all $y\in C$ and $k\in\mathbb{N}$,

$$\| x_{k+1}-Sy\|^2=\|\alpha_k x_k+(1-\alpha_k)Ty_k-Sy\|^2$$
$$\le\alpha_k\| x_k-Sy\|^2+(1-\alpha_k)\| Ty_k-Sy\|^2$$
$$=\alpha_k\| x_k-Sy\|^2+(1-\alpha_k)$$
$$\| Ty_k-Sx_k+Sx_k-Sy\|^2$$
$$\le\alpha_k\| x_k-Sy\|^2+(1-\alpha_k)[\| Ty_k-Sx_k\|^2$$
$$+\| Sx_k-Sy\|^2+2\langle Ty_k-Sx_k,Sx_k-Sy\rangle]$$
$$\le\alpha_k\| x_k-Sy\|^2+(1-\alpha_k)$$
$$[\| Ty_k-x_k\|^2+\| x_k-Sx_k\|^2$$
$$+\| x_k-y\|^2+2\langle x_k-Sx_k,y-Sy\rangle$$
$$+k\| x_k-Sx_k-(y-Sy)\|^2$$
$$+2\langle Ty_k-x_k,Sx_k-Sy\rangle$$
$$+2\langle x_k-Sx_k,Sx_k-Sy\rangle].$$

But we have

$$\| x_k-y\|^2=\| x_k-Sy\|^2+\| Sy-y\|^2$$
$$+2\langle x_k-Sy,Sy-y\rangle,$$

so

$$\| Ty_k-x_k\|^2$$
$$+\| x_k-Sx_k\|^2+2(Ty_k-x_k,x_k-Sx_k)$$
$$+(1-\alpha_k)\left[\| x_k-Sy\|^2+\| Sy-y\|^2+2\langle x_k-Sy,Sy-y\rangle\right]$$
$$+2(1-\alpha_k)\langle x_k-Sx_k,y-Sy\rangle+k(1-\alpha_k)\| x_k-Sx_k\|^2$$
$$+k(1-\alpha_k)\| y-Sy\|^2+2k(1-\alpha_k)\langle x_k-Sx_k,Sy-y\rangle$$
$$+2(1-\alpha_k)\langle Ty_k-x_k,Sx_k-Sy\rangle$$
$$+2(1-\alpha_k)\langle x_k-Sx_k,Sx_k-Sy\rangle.$$

Therefore

$$\| x_{k+1}-Sy\|^2\le a_k\| x_k-Sy\|^2+(1-a_k)\| Ty_k-x_k\|^2$$
$$+(1-a_k)\| x_k-Sx_k\|^2$$
$$+2(1-a_k)\langle Ty_k-x_k,x_k-Sx_k\rangle$$
$$+(1-a_k)(1+k)\| Sy-y\|^2$$
$$+2(1-a_k)\langle x_k-Sy,Sy-y\rangle$$
$$+2(1-a_k)\langle x_k-Sx_k,y-Sy\rangle$$
$$+k(1-a_k)\| x_k-Sx_k\|^2$$
$$+2k(1-a_k)\langle x_k-Sx_k,Sy-y\rangle$$
$$+2k(1-a_k)\langle Ty_k-x_k,Sx_k-Sy\rangle$$
$$+2(1-a_k)\langle x_k-Sx_k,Sx_k-Sy\rangle$$

$$\begin{aligned}&\le \| x_k - Sy \|^2 + \| Ty_k - x_k \|^2 + \| x_k - Sx_k \|^2 \\ &+ 2\langle Ty_k - x_k, x_k - Sx_k \rangle + (1+k) \| Sy - y \|^2 \\ &+ 2\langle x_k - Sy, Sy - y \rangle \\ &+ 2\langle x_k - Sx_k, y - Sy \rangle + \| x_k - Sx_k \|^2 \\ &+ 2\langle x_k - Sx_k, Sy - y \rangle \\ &+ 2\langle Ty_k - x_k, Sx_k - Sy \rangle \\ &+ 2\langle x_k - Sx_k, Sx_k - Sy \rangle.\end{aligned}$$

Summing these inequalities from $k = 1$ to $n$ and dividing by $n$, Since $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0, 1), we have

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n} \| x_{k+1} - Sy \|^2 \le \frac{1}{n}\sum_{k=1}^{n} \| x_{k+1} - Sy \|^2 \\ &+ \frac{1}{n}\sum_{k=1}^{n} \| Ty_k - x_k \|^2 \\ &+ \frac{1}{n}\sum_{k=1}^{n} \| x_k - Sx_k \|^2 + \frac{2}{n}\sum_{k=1}^{n} \langle Ty_k - x_k, x_k - Sx_k \rangle \\ &+ \frac{(1+k)}{n}\sum_{k=1}^{n} \| Sy - y \|^2 + \frac{2}{n}\sum_{k=1}^{n} \langle x_k - Sy, Sy - y \rangle \\ &+ \frac{2}{n}\sum_{k=1}^{n} \langle x_k - Sx_k, y - Sy \rangle + \frac{1}{n}\sum_{k=1}^{n} \| x_k - Sx_k \|^2 \\ &+ \frac{2}{n}\sum_{k=1}^{n} \langle x_k - Sx_k, Sy - y \rangle + \frac{2}{n}\sum_{k=1}^{n} \langle Ty_k - x_k, Sx_k - Sy \rangle \\ &+ \frac{2}{n}\sum_{k=1}^{n} \langle x_k - Sx_k, Sx_k - Sy \rangle.\end{aligned}$$

So we have

$$\begin{aligned}&\frac{1}{n}\| x_{n+1} - Sy \|^2 \le \frac{1}{n}\| x_1 - Sy \|^2 + \frac{1}{n}\sum_{k=1}^{n} \| x_k - Sx_k \|^2 \\ &+ \frac{2}{n}\sum_{k=1}^{n} \| Ty_k - x_k \| \| x_k - Sx_k \| \\ &+ \frac{(1+k)}{n}\sum_{k=1}^{n} \| Sy - y \|^2 + 2\langle U_n - Sy, Sy - y \rangle \\ &+ \frac{2}{n}\sum_{k=1}^{n} \| x_k - Sx_k \| \| y - Sy \| + \frac{1}{n}\sum_{k=1}^{n} \| x_k - Sx_k \|^2 \\ &+ \frac{2}{n}\sum_{k=1}^{n} \| x_k - Sx_k \| \| Sy - y \| \\ &+ \frac{2}{n}\sum_{k=1}^{n} \| Ty_k - x_k \| \| Sx_k - Sy \| \\ &+ \frac{2}{n}\sum_{k=1}^{n} \| x_k - Sx_k \| \| Sx_k - Sy \|. \qquad (14)\end{aligned}$$

#### 3.1.1.4 Step 4

We claim that $\lim_{n\to\infty} \| x_n - Sx_n \| = 0$ and $\lim_{n\to\infty} \| x_n - Ty_n \| = 0$.

From (8) we have,

$$\begin{aligned}\| V_n x - V_n y \|^2 &\le \| x - y \|^2 + 2(1-\beta_n)\langle x - Sx, y - Sy \rangle \\ &- (1-\beta_n)(\beta_n - k)\| x - Sx - (y - Sy) \|^2 \\ &= \| x - y \|^2 + 2(1-\beta_n)\langle x - Sx, y - Sy \rangle \\ &- (1-\beta_n)(\beta_n - k)\| x - Sx \|^2 \\ &- (1-\beta_n)(\beta_n - k)\| y - Sy \|^2 \\ &- 2(1-\beta_n)(\beta_n - k)\langle x - Sx, y - Sy \rangle. \qquad (15)\end{aligned}$$

Putting in (15) $x = x_n$ and $y = z$ so we have,

$$\begin{aligned}\| V_n x_n - z \|^2 &\le \| x_n - z \|^2 + 2(1-\beta_n)\langle x_n - Sx_n, z - Sz \rangle \\ &- (1-\beta_n)(\beta_n - k)\| x_n - Sx_n - (z - Sz) \|^2 \\ &- (1-\beta_n)(\beta_n - k)\| Sz - z \|^2 \\ &- 2(1-\beta_n)(\beta_n - k)\langle x_n - Sx_n, z - Sz \rangle.\end{aligned}$$

Since $z \in F(S) \cap F(T)$,

$$\begin{aligned}\| y_n - z \|^2 = \| V_n x_n - z \|^2 &\le \| x_n - z \|^2 \\ &- (1-\beta_n)(\beta_n - k)\| x_n - Sx_n \|^2 \qquad (16)\end{aligned}$$

We have,

$$\begin{aligned}\| x_{n+1} - z \|^2 &= \| \alpha_n x_n + (1-\alpha_n) Ty_n - z \|^2 \\ &\le \alpha_n \| x_n - z \|^2 + (1-\alpha_n)\| Ty_n - z \|^2 \\ &- \alpha_n (1-\alpha_n)\| x_n - Ty_n \|^2 \\ &\le \alpha_n \| x_n - z \|^2 + (1-\alpha_n)\| y_n - z \|^2 \\ &- \alpha_n (1-\alpha_n)\| x_n - Ty_n \|^2 .\end{aligned}$$

From (16) we have

$$\begin{aligned}\| x_{n+1} - z \|^2 &\le \alpha_n \| x_n - z \|^2 \\ &+ (1-\alpha_n)\| x_n - z \|^2 \\ &- (1-\alpha_n)(1-\beta_n)(\beta_n - k)\| x_n - Sx_n \|^2 \\ &- \alpha_n (1-\alpha_n)\| x_n - Ty_n \|^2 \\ &\le \| x_n - z \|^2 - (1-\alpha_n)(1-\beta_n)(\beta_n - k)\| x_n - Sx_n \|^2 \\ &- \alpha_n (1-\alpha_n)\| x_n - Ty_n \|^2 \qquad (17)\end{aligned}$$

So, $\alpha_n (1-\alpha_n)\| x_n - Ty_n \|^2 \le \| x_n - z \|^2 - \| x_{n+1} - z \|^2$

Since $0 < a < \alpha_n < b < 1$ we have

$$a(1-b)\|x_n - Ty_n\|^2 \le \alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2$$
$$\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0 \text{ as } n \to \infty.$$

Therefore, $\lim_{n\to\infty} \|x_n - Ty_n\| = 0$.

Now we show that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

From (17) we have

$$\|x_{n+1} - z\|^2 \le \|x_n - z\|^2 - (1-\alpha_n)(1-\beta_n)(\beta_n - k)\|x_n - Sx_n\|^2$$
$$-\alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2$$
$$\le \|x_n - z\|^2 - (1-\alpha_n)(1-\beta_n)(\beta_n - k)\|x_n - Sx_n\|^2.$$

So, $(1-\alpha_n)(1-\beta_n)(\beta_n - k)\|x_n - Sx_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2$

Since $0 < a < \alpha_n < b < 1$ and $0 \le k < c < \beta_n < d < 1$ we have

$$(1-b)(1-d)(c-k)\|x_n - Sx_n\|^2$$
$$\le (1-\alpha_n)(1-\beta_n)(\beta_n - k)\|x_n - Sx_n\|^2$$
$$\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0 \text{ as } n \to \infty.$$

Therefore, $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$

### 3.1.1.5 Step 5

We prove that $w \in F(S) \cap F(T)$Since $\{x_n\}$is bounded, $\{u_n\}$ is also bounded. Then there exists a subsequence $\{u_{ni}\}$ of $\{u_n\}$ such that $u_{n_i} \xrightarrow{w} w$ for some $w \in H$. Further, replacing $n$ by $n_i$ in (13), we have

$$\frac{1}{n_i}\|x_{n_i+1} - Ty\|^2 \le \|x_{n_i} - Ty\|^2\ Ty - y\|^2 + 2\langle u_{n_i} - Ty, Ty - y\rangle$$
$$+\frac{1}{n_i}\sum_{k=1}^{n_i}\|Sx_k - x_k\|^2 + \frac{2}{n_i}\sum_{k=1}^{n_i}\|Sx_k - x_k\|\,\|x_k - y|$$
$$+\frac{2}{n_i}\sum_{k=1}^{n_i}\|Sx_k - x_k\|\,\|y - Ty\| + \frac{2}{n_i}\sum_{k=1}^{n_i}\|x_k - Ty_k\|\,\|y - Ty\| \tag{18}$$

From Step 4 we have $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - Ty_n\| = 0$ it yeilds that when, $i \to \infty$ we have

$$\frac{1}{n_i}\sum_{k=1}^{n_i}\|Sx_k - x_k\|^2 \to 0,$$
$$\frac{2}{n_i}\sum_{k=1}^{n_i}\|Ty_k - x_k\|\,\|y - Ty\| \to 0,$$
$$\frac{2}{n_i}\sum_{k=1}^{n_i}\|Sx_k - x_k\|\,\|y - Ty\| \to 0,$$
$$\frac{2}{n_i}\sum_{k=1}^{n_i}\|x_k - Sx_k\|\,\|x_k - y\| \to 0.$$

Also we have, $\lim_{i\to\infty}\frac{1}{n_i}\|x_{i+1} - Ty\|^2 = 0$ and $\lim_{i\to\infty}\frac{1}{n_i}\|x_1 - Ty\|^2 = 0$. Therefore, since $u_{ni}\, w$ as $i \to \infty$, from (18) we have

$$0 \le \|Ty - y\|^2 + 2(w - Ty, Ty - y).$$

Puttingn $y = w$, then

$$0 \le \|Tw - w\|^2 + 2(w - Tw, Tw - w).$$

So, we have

$$0 \le \|Tw - w\|^2$$

It yields that $\|Tw - w\| = 0$ and $Tw - w$. So $w \in F(T)$. (19)

And by replacing $n$ by $n_i$ in (14), we have

$$\frac{1}{n_i}\|x_{n_i+1} - Sy\|^2 \le \frac{1}{n_i}\|x_1 - Sy\|^2 + \frac{1}{n_i}\sum_{k=1}^{n_i}\|x_k - Sx_k\|^2$$
$$+\frac{2}{n_i}\sum_{k=1}^{n_i}\|Ty_k - x_k\|\,\|x_k - Sx_k\|$$
$$+\frac{(1+k)}{n_i}\sum_{k=1}^{n_i}\|Sy - y\|^2 + 2\langle u_{n_i} - Sy, Sy - y\rangle$$
$$+\frac{2}{n_i}\sum_{k=1}^{n_i}\|x_k - Sx_k\|\,\|y - Sy\| + \frac{1}{n_i}\sum_{k=1}^{n_i}\|x_k - Sx_k\|^2$$
$$+\frac{2}{n_i}\sum_{k=1}^{n_i}\|x_k - Sx_k\|\,\|Sy - y\|$$
$$+\frac{2}{n_i}\sum_{k=1}^{n_i}\|Ty_k - x_k\|\,\|Sx_k - Sy\|$$
$$+\frac{2}{n_i}\sum_{k=1}^{n_i}\|x_k - Sx_k\|\,\|Sx_k - Sy\|.$$

Also From step 2 when $i \to \infty$, we have

$$\frac{1}{n_i}\sum_{k=1}^{n_i},\|x_k - Sx_k\|^2 \to, 0,$$
$$\frac{2}{n_i}\sum_{k=1}^{n_i}\|Ty_k - x_k\|\,\|x_k - Sx_k\| \to 0,$$
$$\frac{2}{n_i}\sum_{k=1}^{n_i}\|x_k - Sx_k\|\,\|y - Sy\| \to 0,$$
$$\frac{2}{n_i}\sum_{k=1}^{n_i}\|Ty_k - x_k\|\,\|Sx_k - Sy\| \to 0,$$
$$\frac{2}{n_i}\sum_{k=1}^{n_i}\|x_k - Sx_k\|\,\|Sx_k - Sy\| \to 0.$$

Also we have, $\lim_{i\to\infty}\frac{1}{n_i}\|x_{i+1}-Sy\|^2=0$ and $\lim_{i\to\infty}\frac{1}{n_i}\|x_1-Sy\|^2=0$. Therefore, since $u_{n_i}\xrightarrow{w}w$ as $i\to\infty$ we have

$$0\le(k+1)\|Sy-y\|^2+2\langle w-Sy,Sy-y\rangle.$$

Puttingn $y=w$, so

$$0\le(k+1)\|Sw-w\|^2+2\langle w-Sw,Sw-w\rangle.$$

Therefore, we have

$$0\le((k+1)-2)\|Sw-w\|^2.$$

Since $k+1<2$. It yields that $\|Sw-w\|=0$ and $Sw-w$, so $w\in F(S)$. (20)

Therefore, by (19) and (20), $w\in F(S)\cap F(T)$.

#### 3.1.1.6 Step 6

We show that $\{u_n\}$ converges weakly to $u\in F(T)\cap F(S)$ and $u=\lim_{n\to\infty}Px_n$. Since $F(T)\cap F(S)\ne\phi$, from step 1, we have $\|x_{n+1}-z\|\le\|x_n-z\|$ for all $z\in F(T)\cap F(S)$. So from Lemma 2.2, $\lim_{n\to\infty}Px_n$ exists. Put $u=\lim_{n\to\infty}Px_n$, Then we prove $u_n\xrightarrow{w}u$. Suppose that $\{u_{ni}\}$ be a subsequence of $\{u_n\}$ such that $u_{n_i}\xrightarrow{w}w$. From step 5 $w\in F(T)\cap F(S)$. Now we show $w=u$. Since, $w\in F(T)\cap F(S)$ and by (6), we have

$$\begin{aligned}\langle x_k-Px_k w-u\rangle&=\langle x_k-Px_k,w-Px_k\rangle+\langle x_k-Px_k,Px_k-u\rangle\\&\le\langle x_k-Px_k,Px_k-u\rangle\\&\le\|Px_k-u\|\|x_k-Px_k\|\\&\le\|Px_k-u\|M,\end{aligned}$$

for all $k\in\mathbb{N}$, where $M=\sup\{\|x_k-Px_k\|: k\in\mathbb{N}$. Summing these inequalities from $k=1$ to $n_i$ and dividing by $n_i$, we have

$$\langle u_{n_i}-\frac{1}{n_i}\sum_{k-1}^{n_i}Px_k,w-u\rangle\le\frac{1}{n_i}\sum_{k=1}^{n_i}M\|Px_k-u\|.$$

Since $\lim_{i\to\infty}u_{ni}=w$ and $\lim_{n\to\infty}Px_n=u$, obtain $\langle w-u,w-u\rangle\le0$, so $u=w$.

## 3.2 Theorem

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and $T$ be a nonspreading mapping of $C$ into itself. Suppose thst $z\in C$ and define sequences $\{x_n\}$, $\{y_n\}$ and $\{S_n\}$ as follows: $x_1=x\in C$ and

$$\begin{cases}x_{n+1}=\alpha_n z+(1-\alpha_n)y_n,\\ y_n=\beta_n Tx_n-(1-\beta_n)S_n,\\ S_n=\frac{1}{n}\sum_{k=0}^{n-1}T^k x_n\end{cases}$$

for all $n\in\mathbb{N}$, where $0\le\alpha_n\le1$, $0\le\beta_n\le1$, $\alpha_n\to0$, $\sum_{n=1}^{\infty}\beta_n<\infty$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$. If $F(T)\ne\phi$, then $\{x_n\}$ and $\{S_n\}$ converge strongly to $Pz$, where $P$ is the metric projection of $H$ onto $F(T)$.

### 3.2.1 Proof

Since $F(T)\ne\phi$, $T$ is quasi-nonexpansive. So, for all $u\in F(T)$ and $n\in\mathbb{N}$

$$\begin{aligned}\|S_n-u\|&=\|\frac{1}{n}\sum_{k=0}^{n-1}T^k x_n-u\|\\&\le\frac{1}{n}\sum_{k=0}^{n-1}\|T^k x_n-u\|\\&\le\frac{1}{n}\sum_{k=0}^{n-1}\|x_n-u\|=\|x_n-\text{u}\|.\end{aligned}\tag{21}$$

Therefore,

$$\begin{aligned}\|x_{n+1}-u\|&=\|\alpha_n z+(1-\alpha_n)y_n-u\|\\&\le\alpha_n\|z-u\|+(1-\alpha_n)\|y_n-u\|\\&\le\alpha_n\|z-u\|+(1-\alpha_n)\|\beta_n Tx_n\\&\quad+(1-\beta_n)\text{S}_n-u\|\\&\le\alpha_n\|z-u\|+\beta_n(1-\alpha_n)\|Tx_n-u\|\\&\quad+(1-\alpha_n)(1-\beta_n)\|\text{S}_n-u\|\\&\le\alpha_n\|z-u\|+\beta_n(1-\alpha_n)\|x_n-u\|\\&\quad+(1-\alpha_n)(1-\beta_n)\|x_n-u\|\\&=\alpha_n\|z-u\|+(1-\alpha_n)\|x_n-u\|\end{aligned}$$

Putting $K=\max\{\|z-u\|,\|x_1-u\|$, we have that $\|x_n-u\|\le\text{K}$ for all $n\in\mathbb{N}$. In fact, it is obvious that $\|x_1-u\|\le K$. Suppose that $\|x_k-u\|\le K$ for some $k\in\mathbb{N}$. Then we have

$$\begin{aligned}\|x_{k+1}-u\|&\le\alpha_k\|z-u\|+(1-\alpha_k)\|x_k-u\|\\&\le\alpha_k K+(1-\alpha_k)K.\end{aligned}$$

By induction, we obtain that $\|x_n-u\|\le K$ for all $n\in\mathbb{N}$. So, $\{x_n\}$, $\{y_n\}$ and $\{S_n\}$ are bounded.

Since $\|T^n x_n-u\|\le\|x_n-u\|$, the sequence $\{T^n x_n\}$ is bounded.

Let $n \in \mathbb{N}$ Since $T$ is nonspreading, we have from (2) and for $a \in C$ and $k = 0, 1, 2, ..., n - 1$

$$\|T^{k+1}x_n - Ta\|^2 \le \|T^k x_n - a\|^2 + 2\left\langle T^k x_n - T^{k+1}x_n, a - Ta\right\rangle.$$
$$= \|T^k x_n - Ta + Ta - a\|^2 + 2\left\langle T^k x_n - T^{k+1}x_n, a - Ta\right\rangle$$
$$= \|T^k x_n - Ta\|^2 + \|Ta - a\|^2 + 2\left\langle T^k x_n - T^a, Ta - a\right\rangle + 2\left\langle T^k x_n - T^{k+1}x_n, a - Ta\right\rangle.$$

Summing these inequalities from $k = 0$ to $n - 1$ and dividing by $n$, we have

$$\frac{1}{n}\|T^n x_n - Ta\|^2 \le \frac{1}{n}\|x_n - Ta\|^2 + \|Ta - a\|^2 + 2\langle S_n - Ta, Ta - a\rangle + \frac{2}{n}\langle x_n - T^n x_n, a - Ta\rangle.$$

Since $\{S_n\}$ is bounded, there exists a subsequence $\{S_{n_i}\}$ of $\{S_n\}$ such that $\{S_{n_i}\}$ w $\in C$. Replacing $n$ by $n_i$, we have

$$\frac{1}{n_i}\|T^{n_i} x_{n_i} - Ta\|^2 \le \frac{1}{n_i}\|x_{n_i} - Ta\|^2 + \|Ta - a\|^2 + 2\langle S_{n_i} - Ta, Ta - a\rangle + \frac{2}{n_i}\langle x_{n_i} - T^{n_i} x_{n_i}, a - Ta\rangle.$$

Since $\{x_n\}$ and $\{T^n x_n\}$ are bounded, we have $0 \le \| Ta - a\|^2 + 2\,(w - Ta,\ Ta - a)$ as $i \to \infty$.

Put $a = w$. Then we have

$$0 \le \| Tw - w\|^2 + 2\,(w - Tw,\ Tw - w) = -\| Tw - w\|^2$$

So, $w \in F(T)$. Since

$$\|x_{n+1} - S_n\| = \|\alpha_n z + (1-\alpha_n)y_n - S_n\|$$
$$= \|\alpha_n(z - S_n) + (1-\alpha_n)(y_n - S_n)\|$$
$$\le \alpha_n\|z - S_n\| + (1-\alpha_n)\|y_n - S_n\|$$
$$\le \alpha_n\|z - S_n\| + (1-\alpha_n)\|\beta_n T x_n + (1-\beta_n)S_n - S_n\|$$
$$\le \alpha_n\|z - S_n\| + (1-\alpha_n)\|\beta_n\|Tx_n - S_n\| + (1-\alpha_n)(1-\beta_n)\|S_n - S_n\|.$$

Since $\{S_n\}$ is bounded and $\alpha_n \to 0, \beta_n \to 0$ , we have

$$\lim_{n\to\infty} \|x_{n+1} - S_n\| = 0 \tag{22}$$

Now, we show that $\limsup_{n\to\infty}\left\langle z - Pz, x_{n+1} - Pz\right\rangle \le 0.$ We assume that there exists a subsequence $\{x_{ni+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty}\left\langle z - Pz, x_{n+1} - Pz\right\rangle = \lim_{i\to\infty}\left\langle z - Pz, x_{n_i+1} - Pz\right\rangle,$$

and $x_{n_i+1} \overset{w}{\to} t$. From (22), $S_{n_i} \overset{w}{\to} t$. So from the above argument, we have $t \in F(T)$. Since $P$ is the metric projection of $H$ onto $F(T)$ and from (6) we have,

$$\lim_{i\to\infty}\left\langle z - Pz, x_{n_i+1} - Pz\right\rangle = \left\langle z - Pz, t - Pz\right\rangle \le 0.$$

This implies

$$\lim_{n\to\infty}\left\langle z - Pz, x_{n+1} - Pz\right\rangle \le 0. \tag{23}$$

Since $x_{n+1} - Pz = \alpha_n z + (1 - \alpha_n)y_n - Pz$, and $\alpha_n, \beta_n \in (0,1)$ and From (4), (21) we have,

$$\|x_{n+1} - Pz\|^2 = \alpha_n z + (1-\alpha_n)y_n - Pz\|^2$$
$$= \|\alpha_n(z - Pz) + (1-\alpha_n)(y_n - Pz)\|^2$$
$$\le (1-\alpha_n)^2\|y_n - Pz\|^2 + 2\alpha_n\left\langle z - Pz, x_{n+1} - Pz\right\rangle$$
$$= (1-\alpha_n)^2\|\beta_n Tx_n + (1-\beta_n)S_n - Pz\|^2 + 2\alpha_n\left\langle z - Pz, x_{n+1} - Pz\right\rangle.$$
$$\le (1-\alpha_n)^2\beta_n\|Tx_n - Pz\|^2 + (1-\alpha_n)(1-\beta_n)\|S_n - Pz\|^2 + 2\alpha_n\left\langle z - Pz, x_{n+1} - Pz\right\rangle$$
$$\le \beta_n\|x_n - Pz\|^2 + (1-\alpha_n)\|x_n - Pz\|^2 + 2\alpha_n\left\langle z - Pz, x_{n+1} - Pz\right\rangle.$$

Put $\theta_n = \beta_n\|x_n - Pz\|^2$, $s_n = \|x_n - Pz\|^2$ and $\gamma_n = 2\langle z - Pz, x_{n+1} - Pz\rangle$ in Lemma (2.1). From (23) and $\sum_{k=0}^{\infty}\alpha_n = \infty$, we have $\lim_{n\to\infty}\|x_n - Pz\| = 0$, and from (22) $\lim_{n\to\infty}\|x_n - S_n\| = 0$. Therefore, $\lim_{n\to\infty} S_n = Pz$.

### 3.3 Remark

Theorem 3.2 improves Theorem 1.3 by Kurokawa and Takahashi in the following sense.

It is sufficient to put $\beta_n = 0$ in Theorem 3.2.

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