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# Initial Value Problem of Second Order Intuitionistic Fuzzy Ordinary Differential Equations

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## Abstract

In this research paper, different cases of solution of the second order linear Intuitionistic Fuzzy Ordinary Differential Equations (IFODEs) are discussed. The initial conditions and the coefficients of differential equations are taken as the Generalized Trapezoidal Intuitionistic Fuzzy Numbers (GTrIFNs). The concept for finding the solution of second order linear homogeneous intuitionistic fuzzy ordinary differential equations is discussed in detail.

**Keywords:** Generalized Hukuhara Differentiability, Generalized Trapezoidal Intuitionistic Fuzzy Numbers

## 1. Introduction

Fuzzy sets were introduced by[1] in 1965, along with the membership function. It was generalized into Intuitionistic Fuzzy Sets by[4] in which non-membership function is also considered. Fuzzy Differential Equations (FDEs) is rapidly growing in recent years. Fuzzy Differential equations was introduced by[6] in 1987. FDEs are mostly used in science and engineering. In[3] discussed the second order linear homogeneous fuzzy ordinary differential equations with generalized trapezoidal fuzzy number were taken as the initial condition and coefficients. Differential equations under intuitionistic fuzzy environment were discussed by[4]. In this paper, existence result for intuitionistic fuzzy ordinary second order linear homogeneous differential equation is discussed. The initial conditions and the coefficients are taken as the generalized trapezoidal intuitionistic fuzzy numbers.

## 2. Preliminaries

Definition 2.1[3]: A fuzzy set $A$ is defined by $A\left\{\left(x,\mu_A(x)\right) : x\grave{o}A, \mu_A(x)\grave{o}[0,1]\right\}$. In the pair $\left(x,\mu_A(x)\right)$, the first element belongs to the classic set $A$, the second element $\mu_A(x)$, belongs to the interval [0,1] called membership function.

Definition 2.2 [4]: Let a set $X$ be fixed. An Intuitionistic Fuzzy Set $A$ in $X$ having the form $A=\left\{\left(x,\mu_A(x),\nu_A(x)\right) : x\int X\right\}$ where the $\mu_A(x): X\to[0,1]$ and $\nu_A(x): X\to[0,1]$ define the degree of membership and degree of non-membership respectively of the element $x\int X$ to the set *A*, which is a subset of *X*, for every element of $x\int X$, $0\le\mu_A(x)+\nu_A(x)\le 1$.

Definition 2.3 [4]: A Trapezoidal Intuitionistic Fuzzy Number *A* is a subset of IFN in *R* with following membership function and non-membership function as follows:

and

$$\mu_A(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1} if a_1<x<a_2\\ 1 if a_2\le x\le a_3\\ \dfrac{a_4-x}{a_4-a_3} if a_3<x\le a_4\\ 0 otherwise\end{cases}$$

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Where $a_1^{'} \le a_2 \le a_3 \le a_4^{'}, a_1 \le a_2 \le a_3 \le a_4$ and TrIFN is denoted by
$A_{TrIFN} = \left(a_1, a_2, a_3, a_4; a_1^{'}, a_2, a_3, a_4^{'}\right)$

$$\nu_A(x) = \begin{cases} \frac{a_4 - x}{a_2 - a_1^{'}} if a_1^{'} \le x < a_2 \\ 0 if a_2 \le x \le a_3 \\ \frac{x - a_3}{a_4^{'} - a_3} if a_3 < x \le a_4^{'} \\ 1 otherwise \end{cases}$$

Definition 2.4 [4]: A Generalized Trapezoidal Intuitionistic Fuzzy Number $A$ is a subset of IFN in $R$ with following membership function and non-membership function as follows:

$$\mu_A(x) = \begin{cases} \omega \frac{x - a_1}{a_2 - a_1} if a_1 \le x < a_2 \\ \omega if a_2 \le x \le a_3 \\ \omega \frac{a_4 - x}{a_4 - a_3} if a_3 < x \le a_4 \\ 0 otherwise \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \sigma \frac{a_4 - x}{a_2 - a_1^{'}} if a_1^{'} \le x < a_2 \\ \sigma if a_2 \le x \le a_3 \\ \sigma \frac{x - a_3}{a_4^{'} - a_3} if a_3 < x \le a_4^{'} \\ 0 otherwise \end{cases}$$

Where $a_1^{'} \le a_2 \le a_3 \le a_4^{'}, a_1 \le a_2 \le a_3 \le a_4$ and GTrIFN is denoted by

$$a$$
$$,\left(|1', a_2, a_3, a_4^{'}; \sigma\right)$$
$$a_1, a_2, a_3, a_4; \omega$$

$$A_{GTrIFN} =$$

Definition 2.5 [5]: (Generalized Hukuhara differentiability) let $F : I \to R_F$ and fix $t_0 \in I$ .We say that $F$ is

(i) right differentiable at $t_0$ ,if there exists an element $F'(t_0) \in R_F$ such that for all $h > 0$ sufficiently small, there exists $F(t_0 + h) \ominus F(t_0), F(t_0) \ominus F(t_0 - h)$

and the limits holds $\underset{h \to 0}{limit} \frac{F(t_0 + h) \ominus F(t_0)}{h}$ =

$$\underset{h \to 0}{limit} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0)$$

(or)

(ii) left differentiable at $t_0$ ,if there exists an element $F'(t_0) \in R_F$ such that for all $h > 0$ sufficiently small, there exists $F(t_0) \ominus F(t_0 + h), F(t_0 - h) \ominus F(t_0)$ and the limits holds

$$\underset{h \to 0}{limit} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h \to 0} it \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0)$$

Definition 2.6 [3]: Consider the second order linear homogeneous intuitionistic fuzzy ordinary differential equation $\frac{dz^2}{dt^2} = nz$ with initial condition $z(t_0) = p$ and $\frac{dz(t_0)}{dt} = q$ .The above ODE is called IFODE if any one of the following three concepts holds:

Concept - I Only $p$ and $q$ is a generalized intuitionistic fuzzy number.

Concept - II Only $n$ is a generalized intuitionistic fuzzy number.

Concept - III Both $n$, $p$ and $q$ are generalized intuitionistic fuzzy number.

Definition 2.7 [4]: The solution of intutionistic fuzzy differential equation is of the form

$$[z_1(t,\alpha), z_2(t,\alpha), z_1'(t,\beta), z_2'(t,\beta)]$$

The solution is called strong when

$$\frac{dz_1(t,\alpha)}{d\alpha} > 0,$$
$$\frac{dz_2(t,\alpha)}{d\alpha} < 0 \forall \alpha \in [0,1], z_1(t,\Omega) \leq z_2(t,\Omega)$$
$$\frac{dz_1'(t,\beta)}{d\beta} > 0, \frac{dz_2'(t,\beta)}{d\beta}$$
$$0 \forall \beta \in [0,1], z_1'(t,\psi) \leq z_2'(t,\psi)$$

Otherwise it is weak solution.

## 3. Solution of Second Order Linear Homogeneous IFODE

The solution procedures of first order homogeneous IFODE of Concept-I, Concept-II and Concept-III are described by taking the positive coefficients only. The intuitionistic fuzzy numbers are taken as GTrIFN.

### 3.1 Concept - I

Consider the initial value problem $z''(t) = nz(t), n > 0$

with intial condition $z(t_0) = p$ and $z'(t_0) = q$ Where

$p$ and $q$ are Generalized trapezoidal intuitionistic fuzzy numbers.

Let $p = ((p_1, p_2, p_3, p_4; \Omega_1), (p_1', p_2', p_3', p_4'; \psi_1))$ and $q = ((q_1, q_2, q_3, q_4; \Omega_2), (q_1', q_2', q_3', q_4'; \psi_2))$

Here four types arise.

Type 3.1.1 When $z(t)$ and $z'(t)$ are right differentiable.

Type 3.1.2 When $z(t)$ is right differentiable and $z'(t)$ is left differentiable.

Type 3.1.3 When $z(t)$ is left differentiable and $z'(t)$ is right differentiable.

Type 3.1.4 When $z(t)$ and $z'(t)$ are left differentiable.

Using the concept of Generalized Hukuhara differentiability the type 3.1.1 and type 3.1.4 are same where the type 3.1.2 and type 3.1.3 are same.

### *Solution of type 3.1.1 and type 3.1.4*

Consider:

$$\frac{d^2 z_1(t,\gamma)}{dt^2} = nz_1(t,\gamma)$$
$$\frac{d^2 z_2(t,\gamma)}{dt^2} = nz_2(t,\gamma)$$
$$\frac{d^2 z_1'(t,\delta)}{dt^2} = nz_1'(t,\delta)$$
$$\frac{d^2 z_2'(t,\delta)}{dt^2} = nz_2'(t,\delta)$$

with initial conditions

$$z_1(t_0,\gamma) = p_1 + \frac{\gamma l_p}{\Omega} \; z_1(t_0,\delta) = p_2' + \frac{\delta l_{p'}}{\psi}$$

$$z_2(t_0,\gamma) = p_4 - \frac{\gamma r_p}{\Omega} \; z_2(t_0,\delta) = p_3' + \frac{\delta r_{p'}}{\psi}$$

$$z_1'(t_0,\gamma) = q_1 + \frac{\gamma l_q}{\Omega} \; z_1'(t_0,\delta) = q_2' - \frac{\delta l_{q'}}{\psi}$$

$$z_2'(t_0,\gamma) = q_4 - \frac{\gamma r_q}{\Omega} \; z_2'(t_0,\delta) = q_3' + \frac{\delta r_{q'}}{\psi}$$

Where:

$$l_p = p_2 - p_1; l_q = q_2 - q_1; r_p = p_4 - p_3;$$

$$r_q = q_4 - q_3; l_{p'} = p_2' - p_1'; l_{q'} = q_2' - q_1';$$

$$r_{p'} = p_4' - p_3'; r_{q'} = q_4' - q_3'$$

$$\Omega = min\{;\Omega_1,;\Omega_2\} \text{ and } \psi = min\{\psi_1, \psi_2\}$$

The general solution of equation

$$\frac{d^2 z_1(t,\gamma)}{dt^2} = nz_1(t,\gamma) \text{ is } z_1(t,\gamma) = c_1 e^{\sqrt{n}t} + c_2 e^{-\sqrt{n}t}$$

using initial condition,

$$p_1 + \frac{\gamma l_p}{\Omega} = c_1 e^{\sqrt{n}t_0} + c_2 e^{-\sqrt{n}t_0} \qquad (1)$$

and

$$\sqrt{\phantom{x}} \qquad c\, e^{\sqrt{n}t}$$

$$+ \text{——} =$$

By solving (2) and (3), we get:

$$t$$

$$\sqrt{n}\,(|0-t)$$

$$\left(p_1 + \frac{\gamma l_p}{\Omega}\right) + \frac{1}{\sqrt{n}}\left(q_1 + \frac{\gamma l_q}{\Omega}\right) e$$

$$z_1(t,\gamma) = \frac{1}{2}$$

$$t-t$$

$$-\sqrt{n}\,(|0)$$

$$\left(p_1 + \frac{\gamma l_p}{\Omega}\right) - \frac{1}{\sqrt{n}}\left(q_1 + \frac{\gamma l_q}{\Omega}\right) e$$

$$\frac{1}{2}$$

Similarly, we get:

$$t$$

$$\sqrt{n}\,(|0-t)$$

$$\left(p_4 - \frac{\gamma r_p}{\Omega}\right) + \frac{1}{\sqrt{n}}\left(q_4 - \frac{\gamma r_q}{\Omega}\right) e$$

$$z_2(t,\gamma) = \frac{1}{2}$$

$$t-t$$

$$-\sqrt{n}\,(|0)$$

$$\left(p_4 - \frac{\gamma r_p}{\Omega}\right) - \frac{1}{\sqrt{n}}\left(q_4 - \frac{\gamma r_q}{\Omega}\right) e$$

$$\frac{1}{2}$$

$$t$$

$$\sqrt{n}\,(|0-t)$$

$$\left(p_2 + \frac{\delta l_p}{\psi}\right) + \sqrt{n}\left(q_2 - \frac{\delta l_q}{\psi}\right) e$$

$$z_3(t,\gamma) = \frac{1}{2}$$

$$t-t$$

$$-\sqrt{n}\,(|0)$$

$$\left(p_2 + \frac{\delta l_p}{\psi}\right) - \sqrt{n}\left(q_2 - \frac{\delta l_q}{\psi}\right) e$$

$$\frac{1}{2}$$

$$t$$

$$\sqrt{n}\,(|0-t)$$

$$\left(p_3 + \frac{\delta r_p}{\psi}\right) + \sqrt{n}\left(q_3 + \frac{\delta r_q}{\psi}\right) e$$

$$z_4(t,\gamma) = \frac{1}{2}$$

$$t-t$$

$$-\sqrt{n}\,(|0)$$

$$\left(p_3 + \frac{\delta r_p}{\psi}\right) - \sqrt{n}\left(q_3 + \frac{\delta r_q}{\psi}\right) e$$

$$\frac{1}{2}$$

### *Solution of type 3.1.2 and type 3.1.3*

Consider

$$\frac{d^2 z_2(t,\gamma)}{dt^2} = n z_1(t,\gamma)$$

$$\frac{d^2 z_1(t,\gamma)}{dt^2} = n z_2(t,\gamma)$$

$$\frac{d^2 z_2'(t,\delta)}{dt^2} = n z_1'(t,\delta)$$

$$\frac{d^2 z_1'(t,\delta)}{dt^2} = nz_2'(t,\delta)$$

with same initial conditions.

The general solution is given by:

$$z_1(t,\gamma) = c_1 e^{\sqrt{n}t} + c_2 e^{-\sqrt{n}t} + c_3 cos\sqrt{nt} + c_4 sin\sqrt{nt}$$

$$z_2(t,\gamma) = c_1 e^{\sqrt{n}t} + c_2 e^{-\sqrt{n}t} - c_3 cos\sqrt{nt} - c_4 sin\sqrt{nt}$$

$$z_1'(t,\delta) = d_1 e^{\sqrt{n}t} + d_2 e^{-\sqrt{n}t} + d_3 cos\sqrt{nt} + d_4 sin\sqrt{nt}$$

$$z_2'(t,\delta) = d_1 e^{\sqrt{n}t} + d_2 e^{-\sqrt{n}t} - d_3 cos\sqrt{nt} - d_4 sin\sqrt{nt}$$

$$\left(q_1 + \frac{\gamma l'_q}{\Omega}\right) + \left(q_4 - \frac{\gamma r'_q}{\Omega}\right) + \frac{1}{4}\left(p_1 + \frac{\gamma l'_p}{\Omega}\right) + \left(p_4 - \frac{\gamma r'_p}{\Omega}\right) +$$

$$c_1 = \frac{1}{4}\sqrt{n}$$

$$\left(q_1 + \frac{\gamma l'_q}{\Omega}\right) + \left(q_4 - \frac{\gamma r'_q}{\Omega}\right) - \frac{1}{4}\left(p_1 + \frac{\gamma l'_p}{\Omega}\right) + \left(p_4 - \frac{\gamma r'_p}{\Omega}\right) +$$

$$c_2 = \frac{1}{4}\sqrt{n}$$

$$c_3 = \frac{1}{2}\left\{\left(p_1 + \frac{\gamma l'_p}{\Omega}\right) - \left(p_4 - \frac{\gamma r'_p}{\Omega}\right) cos\sqrt{n}t_0 - \frac{1}{2}\left(q_1 + \frac{\gamma l'_q}{\Omega}\right) + \left(q_4 - \frac{\gamma r'_q}{\Omega}\right)\right\} sin\sqrt{n}t_0$$

$$d_2 = \frac{1}{4}\left[\left(q_2 - \frac{\delta l''_q}{\psi}\right) + \left(q_3 + \frac{\delta r''_q}{\psi}\right) - \sqrt{n}\left[\left(p_2 + \frac{\delta l''_p}{\psi}\right) + \left(p_3 + \frac{\delta r''_p}{\psi}\right)\right]\right] e^{\sqrt{n}t_0}$$

$$d_4 = -\left[\left[\left(q_2 - \frac{\delta l''_q}{\psi}\right) - \left(q_3 + \frac{\delta r''_q}{\psi}\right)\right] sin\sqrt{n}t_0 - \left[ + \left(p_3 + \frac{\delta r''_p}{\psi}\right)\right]\sqrt{n} cos\sqrt{n}t_0\right]$$

## 3.2 Concept - II

Consider the initial value problem $z''(t) = nz(t), n > 0$

with $n' = \left((n_1, n_2, n_3, n_4; \lambda), (n_1', n_2, n_3, n_4'; \varphi)\right)$

intial condition $z(t_0) = p$ and $z'(t_0) = q$ Where $n'$ are Generalized trapezoidal intuitionistic fuzzy numbers.

Here four types arise:

Type 3.2.1 When $z(t)$ and $z'(t)$ are right differentiable.

Type 3.2.2 When $z(t)$ is right differentiable and $z'(t)$ is left differentiable.

Type 3.2.3 When $z(t)$ is left differentiable and $z'(t)$ is right differentiable.

Type 3.2.4 When $z(t)$ and $z'(t)$ are left differentiable.

Using the concept of Generalized Hukuhara differentiability the type 3.2.1 and type 3.2.4 are same where the type 3.2.2 and type 3.2.3 are same.

### *Solution of type 3.2.1 and type 3.2.4*

Consider:

$$\frac{d^2 z_1(t,\gamma)}{dt^2} = n_1(\gamma) z_1(t,\gamma)$$

$$\frac{d^2 z_2(t,\gamma)}{dt^2} = n_2(\gamma) z_2(t,\gamma)$$

$$\frac{d^2 z_1'(t,\delta)}{dt^2} = n_1'(\delta) z_1'(t,\delta)$$

$$\frac{d^2 z_2'(t,\delta)}{dt^2} = n_2'(\delta) z_2'(t,\delta)$$

with initial conditions

$$z_1(t_0,\gamma) = p_1 + \frac{\gamma l'_p}{\Omega} \quad z_1(t_0,\delta) = p_2 + \frac{\delta l''_p}{\psi}$$

$$z_2(t_0,\gamma) = p_4 - \frac{\gamma r'_p}{\Omega} \quad z_2(t_0,\delta) = p_3 + \frac{\delta r''_p}{\psi}$$

$$z_1'(t_0,\gamma) = q_1 + \frac{\gamma l'_q}{\Omega} \quad z_1'(t_0,\delta) = q_2 - \frac{\delta l''_q}{\psi}$$

$$z_2'(t_0,\gamma) = q_4 - \frac{\gamma r'_q}{\Omega} \quad z_2'(t_0,\delta) = q_3 + \frac{\delta r''_q}{\psi}$$

and coefficients

$$n_1(\gamma) = n_1 + \frac{\gamma l'_n}{\lambda} \quad n_1'(\delta) = n_2 - \frac{\delta l'_n}{\varphi}$$

$$n_2(\gamma) = n_4 - \frac{\gamma r'_n}{\lambda} \quad n_2'(\delta) = n_3 + \frac{\delta r'_n}{\varphi}$$

The general solution of equation $\frac{d^2 z_1(t,\gamma)}{dt^2} = n(\gamma) z_1(t,\gamma)$ is

$$z_1(t,\gamma) = c_1 e^{\sqrt{n_1(\gamma)}t} + c_2 e^{-\sqrt{n_2(\gamma)}t}$$

using initial condition,

$$p = c_1 e^{\sqrt{n_1 + \frac{\gamma l_n}{\lambda}} t_0} + c_2 e^{-\sqrt{n_1 + \frac{\gamma l_n}{\lambda}} t_0} \qquad (3)$$

and

$$q = \left( c_1 e^{\sqrt{n_1 + \frac{\gamma l_n}{\lambda}} t_0} - c_2 e^{-\sqrt{n_1 + \frac{\gamma l_n}{\lambda}} t_0} \right) \sqrt{n_1 + \frac{\gamma l_n}{\lambda}} \qquad (4)$$

By solving (3) and (4), we get:

$$-\sqrt{n_1 + \frac{\gamma l_n}{\lambda}}\left(t_0 - t\right)$$

$$\left( p - \frac{q}{\sqrt{n_1 + \frac{\gamma l_n}{\lambda}}} \right) e$$

$$z_1(t,\gamma) = \frac{1}{2}\left[ \left( p + \frac{q}{\sqrt{n_1 + \frac{\gamma l_n}{\lambda}}} \right) e^{\sqrt{n_1 + \frac{\gamma l_n}{\lambda}}(t - t_0)} + \frac{1}{2} \right]$$

$$t - t$$

$$\sqrt{n_2 - \frac{\delta l_n}{\varphi}}\,(|\,0)$$

$$\left( p + \frac{q}{\sqrt{n_2 - \frac{\delta l_n}{\varphi}}} \right) e$$

$$-\sqrt{n_2 - \frac{\delta l_n}{\varphi}}\left(t_0 - t\right)$$

$$\left( p - \frac{q}{\sqrt{n_2 - \frac{\delta l_n}{\varphi}}} \right) e$$

$$z_1'(t,\delta) = \frac{1}{2}$$

Similarly,

$$t - t$$

$$\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}\,(|\,0)$$

$$\left( p + \frac{q}{\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}} \right) e$$

$$-\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}\left(t_0 - t\right)$$

$$\left( p - \frac{q}{\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}} \right) e$$

$$z_2(t,\gamma) = \frac{1}{2}$$

$$t - t$$

$$\sqrt{n_3 + \frac{\delta r_n}{\varphi}}\,(|\,0)$$

$$\left( p + \frac{q}{\sqrt{n_3 + \frac{\delta r_n}{\varphi}}} \right) e$$

$$-\sqrt{n_3 + \frac{\delta r_n}{\varphi}}\left(t_0 - t\right)$$

$$\left( p - \frac{q}{\sqrt{n_3 + \frac{\delta r_n}{\varphi}}} \right) e$$

$$z_2'(t,\delta) = \frac{1}{2}$$

### *Solution of type 3.2.2 and type 3.2.3*

Consider:

$$\frac{d^2 z_1(t,\gamma)}{dt^2} = n_2(\gamma) z_2(t,\gamma)$$

$$\frac{d^2 z_2(t,\gamma)}{dt^2} = n_1(\gamma) z_1(t,\gamma)$$

$$\frac{d^2 z_1^{'}(t,\delta)}{dt^2} = n_2^{'}(\delta) z_2^{'}(t,\delta)$$

$$\frac{d^2 z_2^{'}(t,\delta)}{dt^2} = n_1^{'}(\delta) z_1^{'}(t,\delta)$$

With same initial conditions and coefficients:

$$n_1(\gamma) = n_1 + \frac{\gamma l_n}{\lambda} \quad n_1^{'}(\delta) = n_2 - \frac{\delta l_n}{\varphi}$$

$$n_2(\gamma) = n_4 - \frac{\gamma r_n}{\lambda} \quad n_2^{'}(\delta) = n_3 + \frac{\delta r_n}{\varphi}$$

The general solution is given by:

$$z_1(t,\gamma) = c_1 e^{\sqrt{k(\gamma)}t} + c_2 e^{-\sqrt{k(\gamma)}t} + c_3 sin\sqrt{k(\gamma)}t + c_4 cos\sqrt{k(\gamma)}t$$

$$z_2(t,\gamma) = \sqrt{\frac{n_1(\gamma)}{n_2(\gamma)}}\left(c_1 e^{\sqrt{k(\gamma)}t} + c_2 e^{-\sqrt{k(\gamma)}t} - c_3 sin\sqrt{k(\gamma)}t - c_4 cos\sqrt{k(\gamma)}t\right)$$

$$z_1^{'}(t,\delta) = d_1 e^{\sqrt{k'(\delta)}t} + d_2 e^{-\sqrt{k'(\delta)}t} + d_3 sin\sqrt{k'(\delta)} + d_4 cos\sqrt{k'(\delta)}t$$

$$z_2^{'}(t,\delta) = \sqrt{\frac{n_1^{'}(\delta)}{n_2^{'}(\delta)}}\left(d_1 e^{\sqrt{k'(\delta)}t} + d_2 e^{-\sqrt{k'(\delta)}t} - d_3 sin\sqrt{k'(\delta)} - d_4 cos\sqrt{k'(\delta)}t\right)$$

Where $k(\gamma) = \sqrt{n_1(\gamma) n_2(\gamma)}$; $k'(\delta) = \sqrt{n_1^{'}(\delta) n_2^{'}(\delta)}$

$$c_1 = \frac{1}{4}\left\{p\left(1+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\right) + \frac{q}{\sqrt{n(\gamma)}}\left(1+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\right)\right\} e^{-\sqrt{k(\gamma)}t_0}$$

$$c_2 = \frac{1}{4}\left\{p\left(1+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\right) - \frac{q}{\sqrt{n(\gamma)}}\left(1+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\right)\right\} e^{\sqrt{k(\gamma)}t_0}$$

$$c_3 = \frac{1}{2}\left\{p\left(1-\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\right) sin\sqrt{k(\gamma)}t_0 + \frac{q}{\sqrt{n(\gamma)}}\left(1+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\right)\right\} cos\sqrt{k(\gamma)}t_0$$

$$c_4 = \frac{1}{2}\left\{p\left(1-\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\right) cos\sqrt{k(\gamma)}t_0 - \frac{q}{\sqrt{n(\gamma)}}\left(1+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\right)\right\} sin\sqrt{k(\gamma)}t_0$$

$$d_1 = \frac{1}{4}\left\{q\left(1+\sqrt{\frac{n_2^{'}(\delta)}{n_1^{'}(\delta)}}\right) + p\left(1+\sqrt{\frac{n_2^{'}(\delta)}{n_1^{'}(\delta)}}\right)\sqrt{k'(\delta)}\right\} e^{-\sqrt{k'(\delta)}t_0}$$

$$d_2 = \frac{1}{4}\left\{q\left(1+\sqrt{\frac{n_2^{'}(\delta)}{n_1^{'}(\delta)}}\right) - p\left(1+\sqrt{\frac{n_2^{'}(\delta)}{n_1^{'}(\delta)}}\right)\sqrt{k'(\delta)}\right\} e^{\sqrt{k'(\delta)}t_0}$$

$$d_3 = \frac{1}{2}\left\{q\left(1-\sqrt{\frac{n_2^{'}(\delta)}{n_1^{'}(\delta)}}\right) sin\sqrt{k'(\delta)}t_0 + \left[p\left(1-\sqrt{\frac{n_2^{'}(\delta)}{n_1^{'}(\delta)}}\right)\right] cos\sqrt{k'(\delta)}t_0\right\}\sqrt{k'(\delta)}$$

$$d_4 = \frac{1}{2}\left\{q\left(1-\sqrt{\frac{n_2^{'}(\delta)}{n_1^{'}(\delta)}}\right) cos\sqrt{k'(\delta)}t_0 - \left[p\left(1-\sqrt{\frac{n_2^{'}(\delta)}{n_1^{'}(\delta)}}\right)\right] sin\sqrt{k'(\delta)}t_0\right\}\sqrt{k'(\delta)}$$

## 3.3 Concept - III

Consider the initial value problem $z''(t) = nz(t), n > 0$ with $n = \left((n_1, n_2, n_3, n_4; \lambda), (n_1^{'}, n_2, n_3^{'}, n_4; \varphi)\right)$ initial condition $z(t_0) = p$ and $z'(t_0) = q$ Where $p$ and $q$ are Generalized trapezoidal intuitionistic fuzzy numbers.

Let $p = ((p_1, p_2, p_3, p_4; \Omega_1), (p_1^{'}, p_2, p_3^{'}, p_4; \psi_1))$ and

$$q = ((q_1, q_2, q_3, q_4; \Omega_2), (q_1^{'}, q_2, q_3^{'}, q_4; \psi_2))$$

Here four types arise:

Type 3.3.1 When $z(t)$ and $z^{'(t)}$ are right differentiable.

Type 3.3.2 When $z(t)$ is right differentiable and $z'(t)$ is left differentiable.

Type 3.3.3 When $z(t)$ is left differentiable and $z'(t)$ is right differentiable,

Type 3.3.4 When $z(t)$ and $z'(t)$ are left differentiable.

Using the concept of Generalized Hukuhara differentiability the type 3.3.1 and type 3.3.4 are same where the type 3.3.2 and type 3.3.3 are same.

### *Solution of type 3.3.1 and type 3.3.4*

Consider:

$$\frac{d^2 z_1(t,\gamma)}{dt^2} = n_1(\gamma) z_1(t,\gamma)$$

$$\frac{d^2 z_2(t,\gamma)}{dt^2} = n_2(\gamma) z_2(t,\gamma)$$

$$\frac{d^2 z_1^{'}(t,\delta)}{dt^2} = n_1^{'}(\delta) z_1^{'}(t,\delta)$$

$$\frac{d^2 z_2^{'}(t,\delta)}{dt^2} = n_2^{'}(\delta) z_2^{'}(t,\delta)$$

with initial conditions

$$z_1(t_0,\gamma) = p_1 + \frac{\gamma l_p}{\Omega} \; z_1(t_0,\delta) = p_2 + \frac{\delta l_p}{\psi}$$

$$z_2(t_0,\gamma) = p_4 - \frac{\gamma r_p}{\Omega} \; z_2(t_0,\delta) = p_3 + \frac{\delta r_p}{\psi}$$

$$z_1^{'}(t_0,\gamma) = q_1 + \frac{\gamma l_q}{\Omega} \; z_1^{'}(t_0,\delta) = q_2 - \frac{\delta l_q}{\psi}$$

$$z_2^{'}(t_0,\gamma) = q_4 - \frac{\gamma r_q}{\Omega} \; z_2^{'}(t_0,\delta) = q_3 + \frac{\delta r_q}{\psi}$$

and coefficients.

$$n_1(\gamma) = n_1 + \frac{\gamma l_n}{\lambda} \quad n_1^{'}(\delta) = n_2 - \frac{\delta l_n}{\varphi}$$

$$n_2(\gamma) = n_4 - \frac{\gamma r_n}{\lambda} \quad n_2^{'}(\delta) = n_3 + \frac{\delta r_n}{\varphi}$$

The general solution of equation $\frac{d^2 z_1(t,\gamma)}{dt^2} = n(\gamma) z_1(t,\gamma)$

is $z_1(t,\gamma) = c_1 e^{\sqrt{n_1(\gamma)}t} + c_2 e^{-\sqrt{n_1 + \frac{\gamma l_n}{\lambda}}t}$

By solving we get:

$$p$$

$$\left(|1 + \frac{\gamma l_p}{\Omega}\right) + \frac{q_1 + \frac{\gamma l_q}{\Omega}}{\sqrt{n_1 + \frac{\gamma l_n}{\lambda}}}$$

$$-\sqrt{n_1 + \frac{\gamma l_n}{\lambda}}(t_0 - t)$$

$$\left(p_1 + \frac{\gamma l_p}{\Omega} - \frac{q_1 + \frac{\gamma l_q}{\Omega}}{\sqrt{n_1 + \frac{\gamma l_n}{\lambda}}}\right) e$$

$$\left[|e^{\sqrt{n_1 + \frac{\gamma l_n}{\lambda}}(t - t_0)}\right] + \frac{1}{2}$$

$$z_1(t,\gamma) = \frac{1}{2}$$

Similarly,

$$t - t$$

$$\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}(|0)$$

$$\left(p_4 - \frac{\gamma r_p}{\Omega} + \frac{q_4 - \frac{\gamma r_q}{\Omega}}{\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}}\right) e$$

$$-\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}(t_0 - t)$$

$$\left(p_4 - \frac{\gamma r_p}{\Omega} - \frac{q_4 - \frac{\gamma r_q}{\Omega}}{\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}}\right) e$$

$$z_2(t,\gamma) = \frac{1}{2}$$

$$t - t$$

$$\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}(|0)$$

$$\left(p_4 - \frac{\gamma r_p}{\Omega} + \frac{q_4 - \frac{\gamma r_q}{\Omega}}{\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}}\right) e$$

$$-\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}(t_0 - t)$$

$$\left(p_4 - \frac{\gamma r_p}{\Omega} - \frac{q_4 - \frac{\gamma r_q}{\Omega}}{\sqrt{n_4 - \frac{\gamma r_n}{\lambda}}}\right) e$$

$$z_2(t,\gamma) = \frac{1}{2}$$

$$t - t$$

$$\sqrt{n_2 - \frac{\delta l_n}{\varphi}}(|0)$$

$$\left(p_2 + \frac{\delta l_p}{\psi} + \frac{q_2 - \frac{\delta l_q}{\psi}}{\sqrt{n_2 - \frac{\delta l_n}{\varphi}}}\right) e$$

$$-\sqrt{n_2 - \frac{\delta l_n}{\varphi}}(t_0 - t)$$

$$\left(p_2 + \frac{\delta l_p}{\psi} - \frac{q_2 - \frac{\delta l_q}{\psi}}{\sqrt{n_2 - \frac{\delta l_n}{\varphi}}}\right) e$$

$$z_1^{'}(t,\delta) = \frac{1}{2}$$

$$\frac{t-t}{\sqrt{n_3+\frac{\delta r_n}{\varphi}}(|0)}$$

$$\left(p_3+\frac{\delta r_p}{\psi}+\frac{q_3+\frac{\delta r_q}{\psi}}{\sqrt{n_3+\frac{\delta r_n}{\varphi}}}\right)e$$

$$-\sqrt{n_3+\frac{\delta r_n}{\varphi}}(t_0-t)$$

$$\left(p_3+\frac{\delta r_p}{\psi}-\frac{q_3+\frac{\delta r_q}{\psi}}{\sqrt{n_3+\frac{\delta r_n}{\varphi}}}\right)e$$

$$z_2^{'}(t,\delta)=\frac{1}{2}$$

### *Solution of type 3.3.2 and type 3.3.3*

Consider:

$$\frac{d^2 z_1(t,\gamma)}{dt^2}=n_2(\gamma)z_2(t,\gamma)$$

$$\frac{d^2 z_2(t,\gamma)}{dt^2}=n_1(\gamma)z_1(t,\gamma)$$

$$\frac{d^2 z_1^{'}(t,\delta)}{dt^2}=n_2^{'}(\delta)z_2^{'}(t,\delta)$$

$$\frac{d^2 z_2^{'}(t,\delta)}{dt^2}=n_1^{'}(\delta)z_1^{'}(t,\delta)$$

With same initial conditions and coefficients;

$$n_1(\gamma)=n_1+\frac{\gamma l_n}{\lambda}\quad n_1^{'}(\delta)=n_2-\frac{\delta l_n}{\varphi}$$

$$n_2(\gamma)=n_4-\frac{\gamma r_n}{\lambda}\quad n_2^{'}(\delta)=n_3+\frac{\delta r_n}{\varphi}$$

The general solution is given by:

$$z_1(t,\gamma)=c_1e^{\sqrt{k(\gamma)}t}+c_2e^{-\sqrt{k(\gamma)}t}+c_3sin\sqrt{k(\gamma)}t+c_4cos\sqrt{k(\gamma)}t$$

$$\frac{n_1(\gamma)}{n_2(\gamma)}\left(c_1e^{\sqrt{k(\gamma)}t}+c_2e^{-\sqrt{k(\gamma)}t}-c_3sin\sqrt{k(\gamma)}t-c_4cos\sqrt{k(\gamma)}t\right)$$

$$z_2(t,\gamma)=\sqrt{\phantom{x}}$$

$$z_1^{'}(t,\delta)=d_1e^{\sqrt{k'(\delta)}t}+d_2e^{-\sqrt{k'(\delta)}t}+d_3sin\sqrt{k'(\delta)}+d_4cos\sqrt{k'(\delta)}t$$

$$\frac{n_1^{'}(\delta)}{n_2^{'}(\delta)}\left(d_1e^{\sqrt{k'(\delta)}t}+d_2e^{-\sqrt{k'(\delta)}t}-d_3sin\sqrt{k'(\delta)}-d_4cos\sqrt{k'(\delta)}t\right)$$

$$z_2^{'}(t,\delta)=\sqrt{\phantom{x}}$$

Where $k(\gamma)=\sqrt{n_1(\gamma)n_2(\gamma)};\ k'(\delta)=\sqrt{n_1^{'}(\delta)n_2^{'}(\delta)}$

$$c_1=\frac{1}{4}\left\{\left[\left(p_1+\frac{\gamma l_p}{\Omega}+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\left(p_4-\frac{\gamma r_p}{\Omega}\right)\right)+\sqrt{n(\gamma)}\left(q_1+\frac{\gamma l_q}{\Omega}+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\left(q_4-\frac{\gamma r_q}{\Omega}\right)\right)\right]\right\}e^{-\sqrt{k(\gamma)}t_0}$$

$$c_2=\frac{1}{4}\left\{\left[\left(p_1+\frac{\gamma l_p}{\Omega}+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\left(p_4-\frac{\gamma r_p}{\Omega}\right)\right)-\sqrt{n(\gamma)}\left(q_1+\frac{\gamma l_q}{\Omega}+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\left(q_4-\frac{\gamma r_q}{\Omega}\right)\right)\right]e^{\left(-\sqrt{k(\gamma)}t_0\right)}\right\}$$

$$c_3=\frac{1}{2}\left\{\left[\left(p_1+\frac{\gamma l_p}{\Omega}-\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\left(p_4-\frac{\gamma r_p}{\Omega}\right)\right)\sin\sqrt{k(\gamma)}t_0+\left(q_1+\frac{\gamma l_q}{\Omega}+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\left(q_4-\frac{\gamma r_q}{\Omega}\right)\right)\right]\right\}\cos\sqrt{k(\gamma)}t_0$$

$$c_4=\frac{1}{2}\left\{\left[\left(p_1+\frac{\gamma l_p}{\Omega}-\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\left(p_4-\frac{\gamma r_p}{\Omega}\right)\right)cos\sqrt{k(\gamma)}t_0-\left(q_1+\frac{\gamma l_q}{\Omega}+\sqrt{\frac{n_2(\gamma)}{n_1(\gamma)}}\left(q_4-\frac{\gamma r_q}{\Omega}\right)\right)\right]\right\}sin\sqrt{k(\gamma)}t_0$$

## 4. Application

It has numerous applications and it is mainly useful to find the equilibrium position of the weight when it is suspended from a particular height.

## 5. Conclusion

The linear homogeneous intuitionistic fuzzy second order ordinary differential equations are solved by taking the initial condition and coefficient as generalized trapezoidal intutionistic fuzzy number.

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