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Existence Results for Quasilinear Random Impulsive Neutral Differential Equation with Delay

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Abstract

In this article, we study the existence of mild solutions of quasilinear random impulsive neutral functional differential equations with delay in Banach spaces. Our results are based on the Schauder fixed point approach. Moreover, an example using the main results is shown in this article. **2010 Mathematical Subject Classification:** 34K30, 34K45, 47D06.

Keywords: Fixed Point Theorem, Neutral Differential Equation. Quasilinear, Random Impulse

1. Introduction

In the various areas of science and engineering the obtain analysis, design and assessment of current methods subjected to practical scenarios require deliberate the respective noise random forces in the system attributes.

The consumption of non-random equations that dispense with the random change of the parameters or change by their required ideals knowing how to effect in gross errors. It is more crucial to reflect on the problem as soon as the perturbation term is significantly broadly impulsive in state and it is inherent to anticipate such a spot in biological systems such as processes in circulation of blood, beating of heart and graphs for neural network and autonomous control problems. Therefore, perturbations of impulsive type are added in a real manner.

In this article we find out that existence results of the following equation:

$$\begin{cases} \left[x(t)+g\left(t,x(t)\right)\right]' + B(t,x)x(t) = f\left(t,x(t)\right), \\ \qquad t\in[0,T],\ t\neq \varsigma_k, \\ x_{t_0} = \varphi \\ x(\varsigma_k) = e_k(\upsilon_k)x(\varsigma_k^-), k=1,2,\ldots, \end{cases} \tag{1}$$

where $-B(t,x)$ is the "infinitesimal generator of an analytic semigroup of operators in a Banach space $\mathbb{X}$."

Here, we form the equation (1), $f: L\to\mathbb{X}\to\mathbb{X}$ is uniformly bounded and continuous and the another function $g: L\to\mathbb{X}\to\mathbb{X}$. Consider $L=[0,T], T\in\mathbb{R}$ be any constant. Considering that operator Δ be a set containing elements and τ_k is a random variable form Δ to $D_k \equiv (0,d_k)$ for $k=1,2,\ldots$ where $0<d_k<\infty$. Also consider that υ_i and υ_j are not dependent an any other as $i\neq j$ for i,j Let $e_k: D_k\to\mathbb{R}$, for each $k=1,2,\ldots$, $\varsigma_0=t_0$ and $\varsigma_k=\varsigma_{k-1}+\upsilon_k$ for $k=1,2,\ldots$, here $t_0\in L$ is an arbitrary real number.

Clearly, $t_0=\varsigma_0<\varsigma_1<\ldots<\lim_{k\to\infty}\varsigma_k=\infty; x(\varsigma_k^-)=\lim_{t\to\varsigma_k}x(t)$ according to their paths with the norm $x = sup_{0\leq s\leq T}\left|x(s)\right|$, for each t satisfies $0\leq t\leq T$. In a Banach space $\mathbb{X}$, a norm $||\cdot||$ is given.

Many authors can be solved by the existence of solutions for quasilinear equations in Banach spaces. Balachandran and Park[1] discussed the integro differential equation. Different kinds of quasilinear differential equations solved by Tasio Kato, Vinodkumar *et al.*[2,3,5]. In[6,7] we use the impulsive differential equations. Existence results of random impulsive are discussed about[14]. Anguraj *et al*, proved by the random impulsive pantograph equations[4].

In the past few years, many papers have been concerned to study the existence of solutions for differential systems or differential inclusions with nonlocal conditions[8]. For impulsive differential system or inclusions with nonlocal conditions of order 1 we refer to[11,13]. For impulsive

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differential system or inclusions of fractional order we refer to[9,10,12] and the references therein.

By the above fact we create a fundamental attempt to realise the study of existence and uniqueness conditions for random impulsive quasilinear neutral functional differential evolution equation using fixed point approach.

In Section 2, we introduce some definitions and example theorems. In section 3, we examine the existence results of quasilinear random impulsive neutral differential equations and example.

2. Preliminaries

We look at the Cauchy problem for the quasilinear IVP.

$$\begin{cases} x'(t)+B(t,x)x(t)=f(t,x(t)),\ 0\le s\le t\le T,\\ x(s)=\upsilon \end{cases} \quad (2)$$

with an operator $-B(t,x)$. Assume the following assumptions.

(ђ1) The domain $D\big(B(t,x(t))\big)=D$ of $B(t,x(t)),0\le t\le T$ is dense in $\mathbb{X}$.

(ђ2) For $t\in L$, the resolvent $R(\gamma;B(t,x(t))=\big(\gamma I-B(t,x(t))\big)^{-1}$ of $B(t,x(t))$ exists for all γ with $Re\ \gamma\le 0$ and there is a constant T such that for $Re\ \gamma\le 0$, $t\in L$,

$\|R(\gamma;B(t,x(t))\|\le T[\|\gamma\|+1]^{-1}$

(ђ3) There exists constants G and $0\le\alpha\le 1$ such that $t,s\in L$,

$$B(t,x(t))-B(s,x(t))\le G|t-s|^{\alpha}$$

Theorem 2.1 ([15]). Let $\mathbb{C}\subset\mathcal{Y}$ and $B(t,c)(t,c)\in I\times\mathbb{C}$ be a family of operators satisfying

(ђ1) - (ђ3), there is a unique evolution system $\mathcal{S}_x(t,s)$ on $0\le s\le t\le T$, satisfying

(i) $\mathcal{S}_x(t,s)\le\mathcal{N}_0$ for $0\le s\le t\le T$.

(ii) For $0\le s\le t\le T$, $S_x(t,s):\mathcal{Y}\to D$ and $t\to\mathcal{S}_x(t,s)$ is strongly differentiable in $\mathcal{Y}$. The derivative $\frac{\partial}{\partial t}\mathcal{S}_x(t,s)\in\mathbb{C}(\mathcal{Y})$ and it is strongly continuous on $0\le s\le t\le T$.

Furthermore,

$$\frac{\partial}{\partial t}\mathcal{S}_x(t,s)+B(t,x)\mathcal{S}_x(t,s)=0 \text{ for } 0\le s\le t\le T$$

$$\frac{\partial}{\partial t}\mathcal{S}_x(t,s)=B(t,x)\mathcal{S}_x(t,s)\le\mathcal{N}_0(t-s)^{-1} \text{ and}$$

$B(t,x)\mathcal{S}_x(t,s)B^{-1}(s,x)\le\mathcal{N}_0$, for $0\le s\le t\le T$.

(iii) For every $\varpi\in D,t\in L,\mathcal{S}_x(t,s)\varpi$ is differentiable with respect to $\mathcal{S}$ on $0\le s\le t\le T$,

$$\frac{\partial}{\partial t}\mathcal{S}_x(t,s)\varpi=-\mathcal{S}_x(t,s)B(s,x(s))\varpi.$$

(iv) $\mathcal{S}_x(t,s)$ is strongly continuous for $0\le s\le t\le T$ and

$\mathcal{S}_x(t,\varrho)=\mathcal{S}_x(t,s)\mathcal{S}_x(s,\varrho)$ for $\varrho\le s\le t$,

$\mathcal{S}_x(t,t)=I$.

From the assumption (ђ2) andthat D is dense in $\mathcal{Y}$ given that for each $t\in[0,T],-B(t,x(t))$.

Define the classical solution of (2) as a function $x:[s,T]\to\mathcal{Y}$ which is continuous for $s\le t\le T$ continuously differentiable for $s\le t\le T$, $x(t)\in D$ for $s\le t\le T$, $x(s)=\upsilon$ and $x'(t)+B(t,x)\,x(t)=f(t,x(t))$ holds for $s\le t\le T.x(t)$ as a solution of the initial IVP (2) if it is a classical solution of the problem.

Theorem 2.2 Let $B(t,x)\,x(t)$, $0\le t\le T$ satisfy the condition (ђ1) - (ђ3) and let $\mathcal{S}_x(t,s)$ be the evolution system in Theorem 2.1. If f is Holder continuous on $[0,T]$, then the initial value problem (2) has for every $\upsilon\in\mathcal{Y}$, a unique solution $x(t)$ given by,

$$x(t)=\mathcal{S}_x(t,s)\upsilon+\int_s^t\mathcal{S}_x(t,\upsilon)f(\upsilon,u(\upsilon))d\upsilon.$$

The above theorem's proofs can be found in [11].

A function $x\in C\ (I:\upsilon)$ such that $(t)\in D(B(t,x(t)))$ for $t\in(0,a]$, $x\in\mathbb{C}^{-1}((0,a]:\upsilon)$ and satisfied (2) in $\mathcal{Y}$ is called a classical solution of (2) on I. Moreover $\exists$ a constant $\kappa>0$ s.t for each x, $\upsilon\in\mathbb{C}(I:\mathcal{Y})$ with values in C and every $\varpi\in\mathbb{F}$ we have

$$\mathcal{S}_x(t,s)\varpi-\mathcal{S}_\upsilon(t,s)\varpi\le\kappa\varpi\int_s^t x(\upsilon)-v(\upsilon)d\upsilon.$$

Definition 2.1 ([3])

A semigroup $\{\mathcal{S}(t),t\ge t_0\}$ is said to be uniformly bounded if $\mathcal{S}(t)\le C$ for all $t\ge t_0$

where $C \geq 1$ is some constant. If $C = 1$, then the semigroup is said to be contraction semigroup.

Definition 2.2 ([3])

For a given $T \in (t_0, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B}_T, -\infty < t \leq T\}$ is called a mild solution to system (1) in $(\mathbf{\Omega}, \mathrm{P}, \mathcal{F}_t)$, if

$x(t) \in \mathcal{B}_T$ is $\mathcal{F}_t$ adapted;

$x(t_0 + s) = \psi(s)$ when $s \in (-\infty, 0]$, and

$$\psi x(t) = \Bigg[\sum_{k=0}^{\infty} \prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \mathcal{S}_x(t,0)[\varphi(0) - g(0,\varphi)]$$

$$-\sum_{k=0}^{\infty} \prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} g(t, x_t)$$

$$-\sum_{k=0}^{\infty} \sum_{i=1}^{k} \prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta}$$

$$\int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t,s) B(s,x) g(s, x_s) ds$$

$$-\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t} \mathcal{S}_x(t,s) B(s,x) g(s, x_s) ds$$

$$+\sum_{k=0}^{\infty} \sum_{i=1}^{k} \prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t,s) f(s, x_s) ds$$

$$+\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t} \mathcal{S}_x(t,s) f(s, x_s) ds \Bigg] I_{[\varsigma_k, \varsigma_k + 1]}(t)$$

where $\prod_{j=m}^{n} (.) = 1$ as $m > n$,

$\prod_{j=1}^{k} (b_j(\tau_j)) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) \ldots b_j(\tau_j)$, and

$I_A(.)$ is the index function, i.e.,

$$I_A(t) = \begin{cases} 1, if\ t \in A \\ 0, if\ t \notin A \end{cases}$$

Definition 2.3

Let B be a convex subset of a Banach space E and assume $0\, belongs\, to\, B$. Let $F : B \to B$ be a completely continuous operator and let

$$U(F) = \{x \in B : x = \mathcal{V}Fx\ for\ some\ 0 < \mathcal{V}\}$$

Then either $U(F)$ is unbounded or F has a fixed point.

Lemma 2.4 ([3])

Assume that the following conditions hold:

For $0 < \eta \leq 1, X_\eta$ is a Banach space

For $0 < \eta \leq \beta \leq 1$, the embedding $X_\beta \to X_\iota$ is continuous.

There exists a constant $C_\eta > 0$ depending on $0 < \eta \leq 1$ such that

$$A^\eta \mathcal{S}(t) \leq \frac{C_\eta}{t^{2\eta}}, \text{ t>0}$$

3. Existence Results

In this section, we tend to discuss regarding the solutions for quasilinear differential equation with random impulsive condition by using fractional powers of operators and Schauder fixed point approach.

Let $s > 0$, take $B_s = \{v \in X ; v_{pc} < s\}$ and assume the subsequent conditions.

(d1) There exists a constant $E_\varsigma > 0$ depending on $0 < \varsigma \leq 1$ such that

$$B^\chi \mathcal{S}(t) \leq \frac{E_\chi}{t^{2\chi}}, t > 0$$

(d2) The mapping $g : [t_0, T] \times \hat{E} \to X$ satisfy the there exists a number $\varsigma \in [0,1]$ such that for any x, $y \in \hat{E}, t \in [t_0, T]$ and $g(t, x_t) \in D(B^\varsigma)$ and

$$B^\chi g(t, x_t) - B^\chi g(t, y_t) \leq D_g x - y_t,$$

$$D_g > 0$$

(d3) For all $t \in [t_0, T]$, it follows that $f(t,0), B^\chi g(t,0) \in L^1$ such that

$$f(t,0) \leq P_f$$

$$B^\chi g(t,0) \leq P_g,$$

where P_f, $P_g > 0$ are constants.

(d4) The condition

$$\max_{i,k} \left\{ \prod_{j=1}^{k} e_j(\upsilon_j) \right\} \leq M$$

(d5) $x_0 \in D(B_0^{\delta})$, for any $\lambda > \delta$ and $B_0^{\delta} x_0 < r$

(d6) The function $f(.,\omega)$ is strongly measurable on $[0,T] \forall\, \omega \in \hat{E}$ and $f(t,.) \in E(\hat{E}, X)$ for each $t \in [0,T]$. There existsa non-decreasing function $Y_f \in E(R^+, (0,\infty))$ and $n_f \in H^p([0,T], R^+)$ such that

$$f(t,\omega_t) \le n_f(t) Y_f(\omega)_t, \text{ for all } (t,\omega) \in [0,T] \times \hat{E}$$

(d7) Now, from the properties of $(T(t))_{t\ge 0}$ and f, the Bochner's criterian for integrable functions and the inequality,

$$T(t-s) f(s, x_s + z_s)^2 \le E_0^2 n_f(s) Y_f \frac{(x_s + z_s)^2}{(t-s)^{2\beta}}$$

(d8) The function *f* is continuous and for all $\mathfrak{z} > 0$ with $[0,\mathfrak{z}](\pi,\hat{E}) \subset [0,T] \times \hat{E}$, there exists

$$\mathfrak{Z}_{f,\mathfrak{z}} \in \mathfrak{Z}^q([0,T],, R^+) \text{ such that,}$$

$$f(s,\psi_1) - f(s,\psi_2) \le \mathfrak{Z}_{f,\mathfrak{z}}(s) \psi_1 - \psi_2$$

And suppose the following conditions are:

(G1) The operator $B_0 = B(0, x_0)$ is a closed operator with domain *D*, dense in γ and

$$(\lambda I - B_0)^{-1} \le T_1 [\lambda + 1]^{-1}, \forall\, \lambda \text{ with } Re\lambda \le 0.$$

(G2) For some $\alpha\, belongs\, to [0,1)$ and for any $v\, belongs\, to\, \mathbb{B}_s$, the operator $B(t, B_0^{-\alpha})v$ is well defined on D for all $t\, belongs\, to\, L$. Also, for any $t, v\, belongs\, to\, L$ and for $v, \varpi \in \mathbb{B}_s$,

$$\left[B(t, B_0^{-\alpha} v) - B(v, B_0^{-\alpha} \varpi)\right] B_0^{-1} \le T_2 |t - v|^{\grave{o}} + v - \varpi^p$$

Where $0 < \epsilon \le 1, 0 < \varrho \le 1$.

(G3) For every $t, v \in L$ and $v, \varpi \in \mathbb{B}_s$,

$$g(t, B_0^{-\alpha} v) - g(v, B_0^{-\alpha} \varpi) \le T_3 \left[|t - v|^{\grave{o}} + v - \varpi^p \right]$$

(G4) For every $t, v \in L$ and $v, \varpi \in \mathbb{B}_s$,

$$f(t, B_0^{-\alpha} v) - f(v, B_0^{-\alpha} \varpi) \le T_4 \left[|t - v|^{\grave{o}} + v - \varpi^p \right]$$

(G5) $x_0 \in D(B_0^{\lambda})$, for any $\lambda > \delta$ and $B_0^{\delta} x_0 < r$

Theorem 3.1

If the hypotheses (d1) - (d8) be hold. Then there exists a mild solution of *x*(*t*) of equation (1).

Proof:

Let *T* be an arbitrary number $t_0 < T < +\infty$. Define $\psi : B \to \mathbb{E}$,

$$\psi x(t) = \Bigg[\sum_{k=0}^{\infty} \prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \mathcal{S}_x(t,0)[\varphi(0) - g(0,\varphi)]$$

$$- \sum_{k=0}^{\infty} \prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} g(t, x_t)$$

$$- \sum_{k=0}^{\infty} \sum_{i=1}^{k} \prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta}$$

$$\int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t,s) B(s,x) g(s,x_s) ds$$

$$- \sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t} \mathcal{S}_x(t,s) B(s,x) g(s,x_s) ds$$

$$+ \sum_{k=0}^{\infty} \sum_{i=1}^{k} \prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t,s) f(s,x_s) ds$$

$$+ \sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t} \mathcal{S}_x(t,s) f(s,x_s) ds \Bigg] I_{[\varsigma_k, \varsigma_k + 1]}(t)$$

Step:1 $\boldsymbol{\varnothing}$ is bounded

$$\psi x(t) \le \sum_{k=0}^{\infty} \left[\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \mathcal{S}_x(t,0) \varphi(0) - g(0,\varphi) \right] I_{[\varsigma_k, \varsigma_k + 1]}(t)$$

$$+ \sum_{k=0}^{\infty} \left[\prod_{j=1}^{k} e_j(\upsilon_j) B^{-\varsigma} B^{\varsigma} g(t, x_t) \right] B_0^{\delta}$$

$$I_{[\varsigma_k, \varsigma_k + 1]}(t)$$

$$+ \left[\begin{array}{c} \sum_{k=0}^{\infty} \left(\max_{i,k} \left\{ 1, \prod_{j=1}^{k} e_j(\upsilon_j) \right\} \right) B_0^{\delta} \\ \left(\int_{t_0}^{t} B^{1-\chi}(s,x) \mathcal{S}_x(t,s) B^{\chi} g(s,x_s) ds \right) \end{array} \right]$$

$$I_{[\varsigma_k,\varsigma_k+1]}(t)$$

$$+\left[\begin{array}{c}\sum_{k=0}^{\infty}\left(\max_{i,k}\left\{1,\prod_{j=1}^{k}e_j(\upsilon_j)\right\}\right)B_0^{\delta}\\ \left(\int_{t_0}^{t}\mathcal{S}_x(t,s)f(s,x_s)ds\right)\end{array}\right]$$

$$I_{[\varsigma_k,\varsigma_k+1]}(t)$$

$$\psi x(t)\leq$$

$$+r\left[\max_k\left\{\prod_{j=1}^{k}e_j(\upsilon_j)\right\}\right]B^{-\chi}\left[B^{\chi}g(t,x_t)\boxminus B^{\chi}g(t,0)+B^{\chi}g(t,0)\right]$$

$$+Cr\left[\max_k\left\{\prod_{j=1}^{k}e_j(\upsilon_j)\right\}\right](t-t_0)$$

$$\int_{t_0}^{t}B^{1-\chi}(s,x)B^{\chi}g(s,x_s)ds$$

$$+Cr\left[\max_k\left\{\prod_{j=1}^{k}e_j(\upsilon_j)\right\}\right](t-t_0)$$

$$\int_{t_0}^{t}f(s,x_s)ds$$

By lemma 2.4, (d2) and (d3), the following relation holds:

$$B^{1-\chi}(s,x)B^{\chi}g(s,x_s)$$

$$=B^{1-\chi}(s,x)B^{\chi}g(s,x_s)-B^{\chi}g(s,0)$$

$$+B^{\chi}g(s,0)$$

$$\leq\frac{C_{1-\chi}}{(s,x)^{2(1-\chi)}}\left[D_gx+P_g\right]$$

Then,

$$\psi x(t)\leq MrC\varphi(0)-g(0,\varphi)$$

$$+rMB^{-\chi}\left[D_gx+P_g\right]$$

$$+2Cr\,max\{1,M\}T$$

$$\int_{t_0}^{t}\frac{C_{1-\chi}}{(s,x)^{2(1-\chi)}}\left[D_gx+P_g\right]ds$$

$$+rC\max\{1,M\}T\int_{t_0}^{t}\left[D_fx+P_f\right]ds$$

Taking supremum over $'t'$, we get

$$\psi x(t)\leq E_1+E_2x$$

Where, $E_1=MrC\varphi(0)-g(0,\varphi)$

$$E_2=rMB^{-\grave{u}}\left[D_gx+P_g\right]$$

$$+2Cr\,max\{1,M\}T$$

$$\int_{t_0}^{t}\frac{C_{1-\chi}}{(s,x)^{2(1-\chi)}}\left[D_gx+P_g\right]ds$$

$$+rC\max\{1,M\}T\int_{t_0}^{t}\left[D_fx+P_f\right]ds$$

Where $E_i=1,2,\ldots$ are constants.
Hence ψ is bounded.

Step: 2 ø is continuous

Let $\{x_n\}$ be a convergent sequence of elements of x in B. Then for each $t\,belongs\,to[0,T]$, we have

$$\psi x_n(t)-\psi x(t)$$

$$\leq\sum_{k=0}^{\infty}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^{\delta}\left[\mathcal{S}_{x_n}(t,0)-\mathcal{S}_x(t,0)\right]$$

$$(\varphi(0)-g(0,\varphi))$$

$$+[\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^{\delta}$$

$$\left(\int_{\varsigma_{i-1}}^{\varsigma_i}\mathcal{S}_{x_n}(t,s)B(s,x_n)g(s,x_s)-\mathcal{S}_x(t,s)B(s,x)g(s,x_s)ds\right)$$

$$+\sum_{k=0}^{\infty}B_0^{\delta}\left(\int_{\varsigma_k}^{t}\mathcal{S}_{x_n}(t,s)B(s,x_n)g(s,x_s)-\mathcal{S}_x(t,s)B(s,x)g(s,x_s)ds\right)$$

$$+\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^{\delta}\left(\int_{\varsigma_{i-1}}^{\varsigma_i}\mathcal{S}_{x_n}(t,s)f(s,x_s)-\mathcal{S}_x(t,s)f(s,x_s)ds\right)$$

$$\psi x_n(t)-\psi x(t)$$

$$\leq Mr\left[\mathcal{S}_{x_n}(t,0)-\mathcal{S}_x(t,0)\right]\varphi(0)-g(0,\varphi)$$

$$+\left[\left[\max_{i,k}\left\{1,\prod_{j=1}^{k}e_j(\upsilon_j)\right\}\right]B_0^{\delta}\right.$$

$$\left(\int_{t_0}^{t}\mathcal{S}_{x_n}(t,s)B(s,x_n)g(s,x_s)-\mathcal{S}_x(t,s)B(s,x)g(s,x_s)ds\right)$$

$$+B_0^\delta\left[\max_{i,k}\left\{1,\prod_{j=1}^{k}e_j(\upsilon_j)\right\}\right]$$

$$\left(\int_{t_0}^{t}\mathcal{S}_{x_n}(t,s)f(s,x_s)-\mathcal{S}_x(t,s)f(s,x_s)ds\right)\Bigg]I_{[\varsigma_k,\varsigma_k+1]}(t)$$

$$\le Mr\left[\mathcal{S}_{x_n}(t,0)-\mathcal{S}_x(t,0)\right]\varphi(0)-g(0,\varphi)$$

$$+r\max\{1,M\}\left(\begin{array}{l}\int_{t_0}^{t}\mathcal{S}_{x_n}(t,s)B(s,x_n)g(s,x_s)-\\ \mathcal{S}_x(t,s)B(s,x)g(s,x_s)ds\end{array}\right)I_{[\varsigma_k,\varsigma_k+1]}(t)$$

$$+r\max\{1,M\}\left(\begin{array}{l}\int_{t_0}^{t}\mathcal{S}_{x_n}(t,s)f(s,x_s)-\\ \mathcal{S}_x(t,s)f(s,x_s)ds\end{array}\right)I_{[\varsigma_k,\varsigma_k+1]}(t)$$

$$\psi x_n(t)-\psi x(t)\to 0\, as\, n\to\infty$$

Step: 3ψ is equicontinuous

Let $t_1,t_2\, belongs\, to\left[t_0,T\right]$ and if $t_0<t_1<t_2<T$, then by using hypothesis (d6)- (d7), we have

$$\psi x(t_2)-\psi x(t_1)=$$

$$\left[\sum_{k=0}^{\infty}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta\,\mathcal{S}_x(t_2,0)\left[\varphi(0)-g(0,\varphi)\right]\right.$$

$$-\sum_{k=0}^{\infty}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta g(t_2,x_{t_2})$$

$$-\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta$$

$$\int_{\varsigma_{i-1}}^{\varsigma_i}\mathcal{S}_x(t_2,s)B(s,x)g(s,x_s)ds$$

$$-\sum_{k=0}^{\infty}B_0^\delta\int_{\varsigma_k}^{t_2}\mathcal{S}_x(t_2,s)B(s,x)g(s,x_s)ds$$

$$+\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta\int_{\varsigma_{i-1}}^{\varsigma_i}\mathcal{S}_x(t_2,s)f(s,x_s)ds+$$

$$\left.\sum_{k=0}^{\infty}B_0^\delta\int_{\varsigma_k}^{t_2}\mathcal{S}_x(t_2,s)f(s,x_s)ds\right]I_{[\varsigma_k,\varsigma_k+1]}(t_2)$$

$$-\left[\sum_{k=0}^{\infty}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta\,\mathcal{S}_x(t_1,0)\left[\varphi(0)-g(0,\varphi)\right]\right.$$

$$-\sum_{k=0}^{\infty}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta g(t_1,x_{t_1})$$

$$-\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta$$

$$\int_{\varsigma_{i-1}}^{\varsigma_i}\mathcal{S}_x(t_1,s)B(s,x)g(s,x_s)ds$$

$$-\sum_{k=0}^{\infty}B_0^\delta\int_{\varsigma_k}^{t_1}\mathcal{S}_x(t_1,s)B(s,x)g(s,x_s)ds$$

$$+\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta\int_{\varsigma_{i-1}}^{\varsigma_i}\mathcal{S}_x(t_1,s)f(s,x_s)ds$$

$$\left.+\sum_{k=0}^{\infty}B_0^\delta\int_{\varsigma_k}^{t_2}\mathcal{S}_x(t_1,s)f(s,x_s)ds\right]I_{[\varsigma_k,\varsigma_k+1]}(t_1)$$

$$=\left\{\begin{array}{l}\sum_{k=0}^{\infty}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta\left[\varphi(0)-g(0,\varphi)\right]\\ \left[\mathcal{S}_x(t_2,0)-\mathcal{S}_x(t_1,0)\right]\end{array}\right\}$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

$$+\left[\sum_{k=0}^{\infty}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta\left(g(t_2,x_{t_2})-g(t_1,x_{t_1})\right)\right]$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

$$+\left[\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta\right.$$

$$\int_{\varsigma_{i-1}}^{\varsigma_i}\mathcal{S}_x(t_2,s)B(s,x)g(s,x_s)ds$$

$$\left.+\sum_{k=0}^{\infty}B_0^\delta\int_{\varsigma_k}^{t_2}\mathcal{S}_x(t_2,s)B(s,x)g(s,x_s)ds\right]$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

$$+\left[\begin{array}{l}\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^\delta\int_{\varsigma_{i-1}}^{\varsigma_i}[\mathcal{S}_x(t_2,s)\\ -\mathcal{S}_x(t_1,s)]B(s,x)g(s,x_s)ds\end{array}\right.$$

$$+\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t_1} [\mathcal{S}_x(t_2,s)-\mathcal{S}_x(t_1,s)] B(s,x) g(s,x_s)\,ds$$
$$\left. +\int_{t_1}^{t_2} \mathcal{S}_x(t_2,s)(s,x) g(s,x_s)\,ds \right]$$
$$I_{[\varsigma_k,\varsigma_k+1]}(t_1)$$
$$+\left[\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds\right.$$
$$\left.+\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t_2} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds\right]$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$+\left[\begin{array}{c} \sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \\ \int_{\varsigma_{i-1}}^{\varsigma_i} [\mathcal{S}_x(t_2,s)-\mathcal{S}_x(t_1,s)] f(s,x_s)\,ds \end{array} + \begin{array}{c} \sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t_1} [\mathcal{S}_x(t_2,s)-\mathcal{S}_x(t_1,s)] \\ f(s,x_s)\,ds+\int_{t_1}^{t_2} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds \end{array}\right]$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

Then,

$$\psi x(t_2)-\psi x(t_1) \le I_1+I_2+I_3+I_4+$$
$$I_5+I_6 \quad (2.1)$$

Where,

$$I_1=\left\{\begin{array}{l} \sum_{k=0}^{\infty}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta}\left[\varphi(0)-g(0,\varphi)\right] \\ \left[\mathcal{S}_x(t_2,0)-\mathcal{S}_x(t_1,0)\right] \end{array}\right\}$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$I_2=\left[\sum_{k=0}^{\infty}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta}\left(g(t_2,x_{t_2})-g(t_1,x_{t_1})\right)\right]$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$I_3=$$
$$\left[\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t_2,s) B(s,x) g(s,x_s)\,ds\right.$$
$$\left.+\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t_2} \mathcal{S}_x(t_2,s) B(s,x) g(s,x_s)\,ds\right]$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$I_4=\left[\begin{array}{l} \sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \int_{\varsigma_{i-1}}^{\varsigma_i} [\mathcal{S}_x(t_2,s) \\ -\mathcal{S}_x(t_1,s)] B(s,x) g(s,x_s)\,ds \end{array}\right.$$
$$+\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t_1} [\mathcal{S}_x(t_2,s)-\mathcal{S}_x(t_1,s)] B(s,x) g(s,x_s)\,ds$$
$$\left.+\int_{t_1}^{t_2} \mathcal{S}_x(t_2,s)(s,x) g(s,x_s)\,ds\right]$$
$$I_{[\varsigma_k,\varsigma_k+1]}(t_1)$$
$$I_5=\left[\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds\right.$$
$$\left.+\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t_2} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds\right]$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$I_6=\left[\begin{array}{l} \sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \int_{\varsigma_{i-1}}^{\varsigma_i} [\mathcal{S}_x(t_2,s) \\ -\mathcal{S}_x(t_1,s)] f(s,x_s)\,ds \end{array}\right. +$$
$$\left.\begin{array}{l} \sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t_1} [\mathcal{S}_x(t_2,s)-\mathcal{S}_x(t_1,s)] \\ f(s,x_s)\,ds+\int_{t_1}^{t_2} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds \end{array}\right]$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

Consider,

$$I_1 \le$$
$$\sum_{k=0}^{\infty}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta}\varphi(0)-g(0,\varphi)\mathcal{S}_x(t_2,0)-\mathcal{S}_x(t_1,0)$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$\le max\left\{1,\prod_{j=1}^{k} e_j(\upsilon_j)\right\} B_0^{\delta}\varphi(0)-g(0,\varphi)\mathcal{S}_x(t_2,0)-\mathcal{S}_x(t_1,0)$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$\le \max\{1,M\}rK\mathcal{S}_x(t_2,0)-\mathcal{S}_x(t_1,0)$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$\to 0\, as\, t_2 \to t_1 \quad (2.2)$$

$$I_2 \le$$
$$\left[\sum_{k=0}^{\infty}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^{\delta}\left(g(t_2,x_{t_2})-g(t_1,x_{t_1})\right)\right]$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$\le max\left\{1,\prod_{j=1}^{k}e_j(\upsilon_j)\right\}B_0^{\delta}\left(g(t_2,x_{t_2})-g(t_1,x_{t_1})\right)$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$\le \max\{1,M\}r\left(g(t_2,x_{t_2})-g(t_1,x_{t_1})\right)$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$\to 0\, as\, t_2 \to t_1 \quad (2.3)$$

$$I_3 \le \left[\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^{\delta}\right.$$
$$\int_{\varsigma_{i-1}}^{\varsigma_i}\mathcal{S}_x(t_2,s)B(s,x)g(s,x_s)\,ds$$
$$\left.+\sum_{k=0}^{\infty}B_0^{\delta}\int_{\varsigma_k}^{t_2}\mathcal{S}_x(t_2,s)B(s,x)g(s,x_s)\,ds\right]$$
$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$\le max\left\{1,\prod_{j=1}^{k}e_j(\upsilon_j)\right\}TB_0^{\delta}$$
$$\int_0^{t_2}\mathcal{S}_x(t_2,s)B(s,x)g(s,x_s)\,ds\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$\le \max\{1,M\}rTC\int_0^{t_2}B^{1-\grave{u}}\,(s,x)B^{\grave{u}}\,g(s,x_s)\,ds$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]\le \max\{1,M\}rTC$$
$$\int_0^{t_2}\frac{C_{1-\grave{u}}}{(s,x)^{2(1-\grave{u})}}\left[D_gx+P_g\right]ds\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2)-I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$
$$\to 0\, as\, t_2 \to t_1 \quad (2.4)$$

$$I_4 \le$$
$$\left[\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^{\delta}\int_{\varsigma_{i-1}}^{\varsigma_i}\mathcal{S}_x(t_2,s)-\mathcal{S}_x(t_1,s)\right.$$
$$B(s,x)g(s,x_s)\,ds$$
$$+\sum_{k=0}^{\infty}B_0^{\delta}$$
$$\int_{\varsigma_k}^{t_1}\mathcal{S}_x(t_2,s)-\mathcal{S}_x(t_1,s)B(s,x)g(s,x_s)\,ds$$
$$\left.+\int_{t_1}^{t_2}\mathcal{S}_x(t_2,s)B(s,x)g(s,x_s)\,ds\right]I_{[\varsigma_k,\varsigma_k+1]}(t_1)$$
$$\le \max\{1,M\}rT\int_0^{t_2}\mathcal{S}_x(t_2,s)-\mathcal{S}_x(t_1,s)B^{1-\grave{u}}$$
$$(s,x)B^{\grave{u}}\,g(s,x_s)\,ds$$
$$+r(t_2-t_1)$$
$$\int_{t_1}^{t_2}\mathcal{S}_x(t_2,s)B^{1-\grave{u}}\,(s,x)B^{\grave{u}}\,g(s,x_s)\,ds$$
$$\le \max\{1,M\}rT\int_0^{t_2}\mathcal{S}_x(t_2,s)-\mathcal{S}_x(t_1,s)\frac{C_{1-\grave{u}}}{(s,x)^{2(1-\grave{u})}}\left[D_gx+P_g\right]ds$$
$$+r(t_2-t_1)$$
$$\int_{t_1}^{t_2}\mathcal{S}_x(t_2,s)\frac{C_{1-\grave{u}}}{(s,x)^{2(1-\grave{u})}}\left[D_gx+P_g\right]ds$$
$$\to 0\, as\, t_2 \to t_1 \quad (2.5)$$

$$I_5 \le$$
$$\left[\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k}e_j(\upsilon_j)B_0^{\delta}\right.$$

$$\int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds$$

$$\left.+\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t_2} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds \right]$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2) - I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

$$\leq max\left\{1, \prod_{j=1}^{k} e_j(\upsilon_j)\right\} B_0^{\delta}$$

$$\int_0^{t_2} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2) - I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

$$\leq \max\{1,M\}\, rT \int_0^{t_2} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2) - I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

$$\leq \max\{1,M\}\, rTE_0^2 \int_0^{t_2} \frac{n_f(s)\, y_f x_s}{(t,s)^{2\delta}}\,ds$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2) - I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

$$\leq \max\{1,M\}\, rTE_0^2 \int_0^{t_2} \frac{n_f(s)}{(t,s)^{2\delta}}\,ds$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_2) - I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

$$\to 0\, as\, t_2 \to t_1 \quad (2.6)$$

$$I_6 \leq \left[\begin{array}{l} \sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \\ \int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t_2,s) - \mathcal{S}_x(t_1,s) f(s,x_s)\,ds \end{array}\right.$$

$$\left.+\begin{array}{l} \sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t_1} \mathcal{S}_x(t_2,s) - \mathcal{S}_x(t_1,s) f(s,x_s)\,ds \\ +B_0^{\delta} \int_{t_1}^{t_2} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds \end{array}\right]$$

$$\left[I_{[\varsigma_k,\varsigma_k+1]}(t_1)\right]$$

$$\leq \max\{1,M\}\, rT \int_0^{t_1} \mathcal{S}_x(t_2,s) - \mathcal{S}_x(t_1,s) f(s,x_s)\,ds$$

$$+r(t_2-t_1)$$

$$\int_{t_2}^{t_1} \mathcal{S}_x(t_2,s) f(s,x_s)\,ds$$

$$\leq \max\{1,M\}\, rT \int_0^{t_1} \mathcal{S}_x(t_2,s) - \mathcal{S}_x(t_1,s)\, n_f(s)\, y_f(s) f(s,x_s)\,ds$$

$$+r(t_2-t_1)$$

$$\int_{t_2}^{t_1} \mathcal{S}_x(t_2,s) f(s,x_s)\, n_f(s)\, y_f(s)\,ds$$

$$\leq \max\{1,M\}\, rTE \int_0^{t_1} \mathcal{S}_x(t_2,s) - \mathcal{S}_x(t_1,s)\, n_f(s)\,ds$$

$$+r(t_2-t_1)\, E \int_{t_2}^{t_1} \mathcal{S}_x(t_2,s) f(s,x_s)\, n_f(s)\,ds$$

$$\to 0\, as\, t_2 \to t_1 \quad (2.7)$$

From the Equations (2.2)-(2.7), it follows that the R.H.S of (2.1) approaches to 0 as t_2 *gives* t_1

Hence ψ is equicontinuous.

Step: 4 ψ is uniqueness

Let T be an arbitrary number $t_0 < T < +\infty$. Define $\psi : \mathcal{B} \to \mathcal{B}$ as follows

Now we have to show that ψ is a contraction mapping.

$$\psi x(t) - \psi y(t)$$

$$\leq \left[\sum_{k=0}^{\infty}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t,s) B(s,x)\left[g(s,x_s) - g(s,y_s)\right] ds\right.$$

$$+\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t} \mathcal{S}_x(t,s) B(s,x)\left[g(s,x_s) - g(s,y_s)\right] ds$$

$$+\sum_{k=0}^{\infty}\sum_{i=1}^{k}\prod_{j=1}^{k} e_j(\upsilon_j) B_0^{\delta} \int_{\varsigma_{i-1}}^{\varsigma_i} \mathcal{S}_x(t,s)\left[f(s,x_s) - f(s,y_s)\right] ds$$

$$\left.+\sum_{k=0}^{\infty} B_0^{\delta} \int_{\varsigma_k}^{t} \mathcal{S}_x(t,s)\left[f(s,x_s) - f(s,y_s)\right] ds\right] I_{[\varsigma_k,\varsigma_k+1]}(t)$$

$$\leq \left[max\left\{1, \prod_{j=1}^{k} e_j\left(\upsilon_j\right)\right\} B_0^{\delta} \right.$$

$$\int_{t_0}^{t} \mathcal{S}_x(t,s) B(s,x)\left[g(s,x_s)-g(s,y_s)\right] ds$$

$$I_{[\varsigma_k,\varsigma_k+1]}(t)$$

$$\leq \max\{1,M\} rTC\frac{C}{t^2}\int_{t_0}^{t} \mathfrak{Z}_{g,\mathfrak{z}} x_s - y_s ds$$

$$+\max\{1,M\} rTC\int_{t_0}^{t} \mathfrak{Z}_{f,\mathfrak{z}} x_s - y_s ds$$

$$\leq \left[\begin{array}{l} \max\{1,M\} rT\frac{C^2}{t^2}\mathfrak{Z}_{g,\mathfrak{z}} \\ +\max\{1,M\} rTC\mathfrak{Z}_{f,\mathfrak{z}} \end{array} \right] x-y$$

$$\leq \Xi x - y$$

With $\Xi = \max\{1,M\} rT\frac{C^2}{t^2}\mathfrak{Z}_{g,\mathfrak{z}} + \max\{1,M\} rTC\mathfrak{Z}_{f,\mathfrak{z}}$

Then we can suitable $0 < T_1 < T$ sufficient small such that $\Xi < 1$, and hence ψ is a contraction on $\mathcal{B} \to \mathcal{B}$.

Lastly, by the Schauder's fixed point technique, ψ has a fixed point and which is a mild solution of the problem (1).

3.1 Example

Consider the following nonlinear parabolic random impulsive differential equation

In a barrel $\mathcal{Q}_T = \rho\times(0,T)$ with coefficients in $\overline{\mathcal{Q}_T}$, where ρ is a bounded domain in $\mathbb{R}^n, \partial\rho$ the boundary of ρ, v is the outward normal. Here the parabolicity means that for any vector $z \neq 0$ and for arbitrary values of $y,t,x,Dx,\ldots,D^{2n-1}x$,

$$(-1)^n \left\{ \sum_{|\alpha|=2n} a_\alpha\left(y,t,Dx,\ldots,D^{2n-1}x\right) z^\alpha \right\} \geq T|z|^{2m}, > 0$$

Assume that υ_k is the random variable defined on $D_k \equiv (0,d_k)$ for $k=1,2,\ldots$, where $0 < d_k < +\infty$. Moreover assume that υ_i & υ_j are not dependent of each other as $i \neq j$ for $i,j = 1,2,\ldots; \varsigma_0 = t_0, \varsigma_k = \varsigma_{k-1} + \upsilon_k$ for k = 1, 2,..., and $\max_{i,k} \prod_{j=1}^{k} q_k\left(\upsilon_j\right) < \infty$. Here $t_0 \in \mathbb{R}_\chi$ is an arbitrarily given real number.

If $x_0(y) \in T^{2n-1}(\bar{\rho})$, then $B_0 x = \sum_{|\alpha|=2n} a_\alpha\left(y,t,Dx,\ldots,D^{2n-1}x\right) D^\alpha x$ is a strongly eliptic operator with continuous coefficients. So (G1) holds. Let us take $\mathcal{Y}$ to be $\mathcal{L}^p(\rho) < p < \infty$. Then B_0^{-1} maps bounded subsets of $\mathcal{L}^p(\rho)$ in to bounded subsets of $\mathcal{W}^{2n,p}(\rho)$, so it is a completely continuous operator in $\mathcal{L}^p(\rho)$. Further, if $\frac{2n-1}{2n} < \alpha < 1$, then $\left|D^\beta B_0 - \alpha x\right|_{0,p}^{\rho} \leq T|x|_{0,p}^{\rho}$, $0 \leq |\beta| \leq 2n-1$, where T depends simplest on a sure at the coefficients B_0 on a module of robustellipticity and on a modulus of continuity of the main coefficients. Hence the norm is described as,

$$|x|_{0,p}^{\rho} = \left\{ \sum_{|\alpha|\leq j} \int_{\rho} \left|D^\alpha x(y)\right|^p dx \right\}^{\frac{1}{p}}$$

For any positive integer j and a real number p, $1 \leq p < \infty$. After that if f and a_α are continuously differentiable in all variables, then (G2)-(G3) hold with Þ = v = 1. Hence there exist fundamental operator solution $\mathcal{S}_y(t,s)$ for Equation (3). The nonlinear function f fulfilled the (G3)-(G5). Hence by the theorem 3.1 there exists a local solution of (3).

4. References

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