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On 3-dimensional Extended Index Matrices

Velichka Traneva[1*], Krassimir Atanassov[2] and R. Parvathi[3]

[1]Professor Asen Zlatarov University, Professor Yakimov Blvd, Bourgas 8000, Bulgaria; veleka13@gmail.com
[1,2]Department of Bioinformatics and Mathematical Modelling, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences, Acad. G. Bonchev Street, Bl. 105, Sofia-1113, Bulgaria; krat@bas.bg
[3]Department of Mathematics, Vellalar College for Women, Erode – 638012, Tamilnadu, India; paarvathis@rediffmail.com

Abstract

An extension of the concept of an Index Matrix (IM), called extended index matrices (EIMs) is introduced. Different operations over EIMs are also introduced.

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1. Introduction

In 1984, Prof. K.T. Atanassov introduced the concept of *index matrix* (IM) [1, 2]. During the next 25 years, some of its properties were studied, but in general it was only used as an auxiliary tool for describing of other mathematical objects. In the beginning of the year, Prof. K. T. Atanassov collected his research over IM and published the book [7]. In it, a few extensions of the IMs are defined. Here, a new extension has been introduced that generalizes two previous extensions:Extended IMs (EIMs) and 3-dimensional IMs (3D-IMs). The following are the assumptions and notations used.

(i) $\mathcal{J}$ is fixed set of indices
(ii) $\mathcal{R}$ be the set of the real numbers
(iii) $(K, L \subset \mathcal{J})$ are index sets

2. Basic Definitions

Definition [1, 2, 7]: Consider a fixed set of indices ($\mathcal{J}$)and a set of real numbers ($\mathcal{R}$) . By IM we mean the object:

$$[K, L, \{f_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \cdots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \cdots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \cdots & a_{k_2,l_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \cdots & a_{k_m,l_n} \end{array},$$

where, $K = \{k_1, k_2, \ldots k_m\}$, $L = \{l_1, l_2, \ldots l_n\}$, for $1 \le i \le m$ and $1 \le j \le n$: $a_{k_i,l_j} \in$ R.

In [5, 7], the IM is extended to the following form: for $\mathcal{J}$,

$$\mathcal{J}^n = \{\langle i_1, i_2, \ldots i_n\rangle | (\forall j : 1 \le j \le n)(i_j \in \mathcal{J})\}$$

and

$$J^* = \bigcup_{1 \le n \le \infty} J^n$$

Let $\mathcal{X}$ be a fixed set of objects. In particular, they can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions or predicates, etc. Let the operators $\circ,*$: $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be fixed. An EIM with index sets K and $L (K, L \subset \mathcal{J}^*)$ and elements from the set $\mathcal{X}$ is an object of the form:

$$[K, L, \{a_{k_i}, l_j\}] \equiv \begin{array}{c|ccccc} & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1,l_1} & \cdots & a_{k_1,l_j} & \cdots & a_{k_1,l_n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_i & a_{k_i,l_1} & \cdots & a_{k_i,l_j} & \cdots & a_{k_i,l_n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_m & a_{k_m,l_1} & \cdots & a_{k_m,l_j} & \cdots & a_{k_m,l_n} \end{array},$$

where, $K = \{k_1, k_2, \ldots k_m\}$, $L = \{l_1, l_2, \ldots l_n\}$ for $1 \le i \le m$, and $1 \le j \le n$: $a_{k_i,l_j} \in \mathcal{X}$. In [6,7,9] the concept of a 3D-IM is introduced, as follows. Let $\mathcal{J}$ be a fixed set of indices and

*Author for correspondence

$\mathcal{X}$ be a fixed set of objects. Following [6], we call "3D-IM" with index sets K, L and $H(K, L, H \subset \mathcal{J})$, the object:

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]$$

$$\equiv \left\{ \begin{array}{c|ccccc} h_g & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1,l_1,h_g} & \vdots & a_{k_1,l_j,h_g} & \cdots & a_{k_1,l_n,h_g} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_m & a_{k_m,l_1,h_g} & \cdots & a_{k_m,l_j,h_g} & \cdots & a_{k_m,l_n,h_g} \end{array} \middle| h_g \in H \right\}$$

$$\equiv \left\{ \begin{array}{c|ccccc} h_1 & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1,l_1,h_1} & \vdots & a_{k_1,l_j,h_1} & \cdots & a_{k_1,l_n,h_1} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_m & a_{k_m,l_1,h_1} & \cdots & a_{k_m,l_j,h_1} & \cdots & a_{k_m,l_n,h_1} \end{array} \right.,$$

$$\begin{array}{c|ccccc} h_2 & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1,l_1,h_2} & \vdots & a_{k_1,l_j,h_2} & & a_{k_1,l_n,h_2} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_m & a_{k_m,l_1,h_2} & \cdots & a_{k_m,l_j,h_2} & \cdots & a_{k_m,l_n,h_2} \end{array} \ldots,$$

$$\left. \begin{array}{c|ccccc} h_f & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1,l_1,h_f} & \vdots & a_{k_1,l_j,h_f} & \cdots & a_{k_1,l_n,h_f} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_m & a_{k_m,l_1,h_f} & \cdots & a_{k_m,l_j,h_f} & \cdots & a_{k_m,l_n,h_f} \end{array} \right\}$$

where, $= K = \{k_1, k_2, \ldots k_m\}$, $L = \{l_1, l_2, \ldots l_n\}$, $H = \{h_1, h_2, \ldots h_f\}$ for $1 \le i \le m$, *and* : $1 \le j \le n$, $1 \le g \le f$: $a_{k_i,l_j,h_g} \in \mathcal{X}$. The intuitionistic fuzzy pair (IFP) is an object with the form $\langle a, b\rangle$, where, $a, b \in [0,1]$ and $a + b \le 1$ that is used as an evalution of some object or process. Where a and b denote degrees of membership (validity or correctness) and non- membership (non-validity or non-correctness) respectively. Consider two IFPs $x = \langle a, b\rangle$ and $y = \langle b, d\rangle$. The following relations have been defined in [8]:

$$x < y \text{ iff } a < c \text{ and } b > d$$

$$x \le y \text{ iff } a < c \text{ and } b \ge d$$

$$x = y \text{ iff } a = c \text{ and } b = d$$

$$x \ge y \text{ iff } a \ge c \text{ and } b \le d$$

$$x > y \text{ iff } a > c \text{ and } b < d$$

Similarly, the operations conjunction and disjunction can be defined as follows:

$$\neg x = \langle a, b\rangle$$

$$x \wedge y = \langle min(a, c), min(b, d)\rangle$$

$$x \vee y = \langle max(a, c), min(b, d)\rangle$$

$$x + y = \langle a + c - a{\cdot}c, b{\cdot}d\rangle$$

$$x{\cdot}y = \langle a{\cdot}c, b + d - b{\cdot}d\rangle$$

$$x@y = \left\langle \frac{a+c}{2}, \frac{b+d}{2} \right\rangle$$

In [4], definitions of 138 implication operations and 34 negation operation are given – the simplest one is given above.

3. Definition of a 3D-Extended Index Matrix (3D-EIM)

Let X be a fixed set of some objects (real numbers, the number 0 or 1, logical variables, propositions or predicates, IFPs, etc.). Now, define a "3D-Extended Index Matrix" (3D-EIM) with index sets K, L and $H(K, L, H \subset \mathcal{J}^*)$ and elements from the set X:

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]$$

$$= \left\{ \begin{array}{c|ccccc} h_g & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1,l_1,h_g} & \vdots & a_{k_1,l_j,h_g} & \cdots & a_{k_1,l_n,h_g} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_i & a_{k_i,l_1,h_g} & \cdots & a_{k_i,l_j,h_g} & \cdots & a_{k_i,l_n,h_g} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_m & a_{k_m,l_1,h_g} & \cdots & a_{k_m,l_j,h_g} & \cdots & a_{k_m,l_n,h_g} \end{array} \middle| h_g \in H \right\}$$

where, $K = \{k_1, k_2, \ldots k_m\}$, $L = \{l_1, l_2, \ldots l_n\}$, $H = \{h_1, h_2, \ldots h_f\}$ for $1 \le i \le m$, and : $1 \le j \le n$, $1 \le g \le f$: $a_{k_i,l_j,h_g} :\in \mathcal{X}$.

4. Operations over 3D-EIMs

Let $3D - EIM_{\mathcal{R}}$ be the set of all 3D-EIMs with elements being real numbers, $3D - EIM_{\{0,1\}}$ be the set of all (0, 1)-3D-EIMs with elements being 0 or 1, 3D – IMP be the set of all 3D-EIMs with elements – predicates and $3D - EIM_{IFP}$ be the set of all 3D-EIMs with elements – IFPs.

Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{U}$ be fixed sets. Let the operations " $*$ " and " $\circ$ " be defined so that:

$* : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ and $\circ : \mathcal{Z} \times \mathcal{Z} \to \mathcal{U}$. Let the index set $\mathcal{J}^*$ be given. We will define some operations over the 3D-EIMs A and B

4.1 Transposition

As discussed in [5, 7], there are 2 (= 2!) EIMs, related to this operation: the standard EIM and its transposed EIM. Now, for 3D-EIMs, there are 6 (= 3!) cases: the standard 3D-IM and five different transposed 3D-IMs. By analogy with [7], it is shown that the following are the geometrical and analytical forms of the separate transposed 3D-EIMs,

[1,2,3]-transposition (identity)

$$\left(\begin{smallmatrix} & H \\ K & L \end{smallmatrix}\right)^{[1,2,3]} = \begin{smallmatrix} & H \\ K & L \end{smallmatrix}$$

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[1,2,3]} = [K, L, H, \{a_{k_i,l_j,h_g}\}]$$

[1,3,2]-transposition

$$\left(\begin{smallmatrix} & H \\ K & L \end{smallmatrix}\right)^{[1,2,3]} = \begin{smallmatrix} & L \\ K & H \end{smallmatrix}$$

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[1,2,3]} = [K, L, H, \{a_{k_i,h_g,l_j}\}]$$

[2,1,3]-transposition

$$\left(\begin{smallmatrix} & H \\ K & L \end{smallmatrix}\right)^{[2,1,3]} = \begin{smallmatrix} & H \\ L & K \end{smallmatrix}$$

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[2,1,3]} = [L, K, H, \{a_{l_j,k_i,h_g}\}]$$

[2,3,1]-transposition

$$\left(\begin{smallmatrix} & H \\ K & L \end{smallmatrix}\right)^{[2,3,1]} = \begin{smallmatrix} & K \\ L & H \end{smallmatrix}$$

$$[K, L, H\ \{a_{k_i,l_j,h_g}\}]^{[2,3,1]} = [L, H, K\{a_{l_j,h_g,k_i}\}]$$

[3,1,2]-transposition

$$\left(\begin{smallmatrix} & H \\ K & L \end{smallmatrix}\right)^{[3,1,2]} = \begin{smallmatrix} & L \\ H & K \end{smallmatrix}$$

$$[K, L, H\ \{a_{k_i,l_j,h_g}\}]^{[3,2,1]} = [H, K, L\{a_{h_g,k_i,l_j}\}]$$

[3,2,1]-transposition

$$\left(\begin{smallmatrix} & H \\ K & L \end{smallmatrix}\right)^{[3,2,1]} = \begin{smallmatrix} & K \\ H & L \end{smallmatrix}$$

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[3,2,1]} = [H, L, K\ \{a_{h_g,l_j,k_i}\}]$$

4.2 Addition [3]

$$A \oplus_{(\circ)} B = [K \cup P, L \cup Q, H \cup R, \{C_{t_u,v_w,x_y}\}]$$

where

$$\{C_{t_u,v_w,x_y}\} = \begin{cases} a_{k_i,l_j,h_g}, & \text{if } t_u = k_i \in K, v_w = l_j \in L \text{ and } x_y = h_g \in H - R \\ & \text{or } t_u = k_i \in K, v_w = l_j \in L - Q \text{ and } x_y = h_g \in H \\ & t_u = k_i \in K, v_w = l_j \in L \text{ and } x_y = h_g \in H - R \\ b_{p_r,q_s,e_d}, & \text{if } t_u = p_r \in P, v_w = q_s \in Q \text{ and } x_y = e_d \in R - H \\ & \text{if } t_u = p_r \in P, v_w = q_s \in Q - L \text{ and } x_y = e_d \in R \\ & \text{if } t_u = p_r \in P - K, v_w = q_s \in Q \text{ and } x_y = e_d \in R \\ a_{k_i,l_j,h_g} \circ b_{p_r,q_s,e_d}, & \text{if } t_u = k_i = p_r \in K \cap P, v_w = l_j = q_s \in L \cap Q \\ & \text{and } x_y = h_g = e_d \in H \cap R \\ 0 & \text{for } A, B \in 3D - EIM_R \text{ other wise} \\ & \text{or } A, B \in 3D - EIM_{\{0,1\}} \\ F & \text{for } A, B \in 3D - EIM_P \\ \{0,1\} & \text{for } A, B \in 3D - EIM_{IFP} \end{cases}$$

Where, here and below F means "false" and

$$\circ \in \begin{cases} \{"+", "\times", "\max", "\min"\}, & \text{if } A \in 3D - EIM_R \\ \{"\max", "\min"\} & \text{if } A \in 3D - EIM_{\{0,1\}} \\ \{"\wedge", "\vee", "\rightarrow", "\equiv"\}, \text{if } A \in 3D - EIM_P & \text{or } A \in 3D - EIM_{IFP} \end{cases}$$

4.3 Termwise Multiplication [3]

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, H \cap R, \{C_{t_u,v_w,x_y}\}]$$

Where, $\{C_{t_u,v_w,x_y}\}] = a_{k_i,l_j,h_g} \circ b_{p_r,q_s,e_d}$ for $t_u = k_i = p_r \in K \cap P$, $v_w = l_j = q_s \in L \cap Q$ *and* $x_y = h_g = e_d \in H \cap R$. Here, "$\circ$" is as above.

4.4 Multiplication [3]

This operation is related to the operation "transposition". There are six different operations "multiplication". The first (standard) multiplication is

$$A \underset{(\circ,*)}{\odot} B = A \underset{(\circ,*)}{\overset{[1,2,3]}{\odot}} [K \cup Q, L \cup P, H \cup R, \{C_{t_u,v_w,x_y}\}]$$

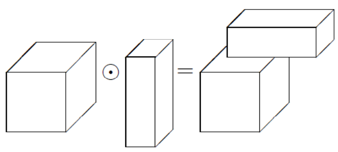
where

$$\{C_{t_u,v_w,x_y}\}=\begin{cases} a_{k_i,l_j,h_g}, & \text{if } t_u=k_i\in K, v_w=l_j\in L-P-Q \text{ and } x_y=h_g\in H \\ & \text{or } t_u=k_i\in K-P-Q, v_w=l_j\in L \text{ and } x_y=h_g\in H \\ b_{p_r,q_s,e_d}, & \text{if } t_u=p_r\in P, v_w=q_s\in Q-L-K \text{ and } x_y=e_d\in R \\ & \text{or } t_u=p_r\in P, v_w=q_s\in Q-L-K \text{ and } x_y=e_d\in R \\ i_j^{\circ} a_{k_i,l_j,h_g} * b_{p_r,q_s,e_d}, & \text{if } t_u==\in K, v_w=q_s\in Q \\ & \text{and } x_y=h_g=e_d\in H\cap R \\ 0 & \text{for } A,B\in 3D-EIM_R \text{ other wise} \\ & \text{or } A,B\in 3D-EIM_{\{0,1\}} \\ F & \text{for } A,B\in 3D-EIM_P \\ \{0,1\} & \text{for } A,B\in 3D-EIM_{IFP} \end{cases}$$

where

$$(\circ,*)\in\begin{cases} \{(+,\times),\ \text{"(max, min)"}\}, & \text{if } A\in 3D-EIM_R \\ \{\text{"(max, min)"},\text{"(min, max)"},\} & \text{if } A\in 3D-EIM_{\{0,1\}} \\ \{\text{"}(\wedge,\vee)\text{"},\text{"}(\wedge,\vee)\text{"}\},\ \text{if } A\in 3D-EIM_P & \text{or } A\in 3D-EIM_{IFP} \end{cases}$$

The geometrical interpretation of this operation is



Let $[x, y, z]$ be a permutation of triple $[1,2,3]$. Operation $[x, y, z]$-multiplication is defined by:

$$A \underset{(\circ,*)}{\overset{[x,y,z]}{\odot}} B = A \underset{(\circ,*)}{\odot} B^{[x,y,z]}$$

For operations "addition", "termwise multiplication" and "multiplication", in the case of $3D-EIM_{\mathcal{R}}$, $\mathcal{X}=\mathcal{Y}=\mathcal{Z}=\mathcal{R}$; in the case of $(0,1)-3D\ EIMs$, $\mathcal{X}=\mathcal{Y}=\mathcal{Z}=\{', \infty\}$ in the case of $3D-IM_{\mathcal{P}}$, $\mathcal{X}=\mathcal{Y}=\mathcal{Z}$, is a set of logical variables, propositions or predicates; in the case of $3D-IM_{\mathcal{IFP}}$, $\mathcal{X}=\mathcal{Y}=\mathcal{Z}$ is a set of IFPs.

4.5 Structural Subtraction [3]

$$A\ominus B=[K-P, L-Q, H-R, \{C_{t_u,v_w,x_y}\}],$$

where – is the set-theoretic difference operation and

$$C_{t_u,v_w,x_y}=a_{k_i,l_j,h_g} \text{ for } t_u=k_i\in K-P, v_w=l_j\in L-Q \text{ and}$$

$$x_y=h_g\in H-R$$

4.6 Multiplication with a Constant

$$\alpha.A=[K, L, H, \{\alpha, a_{k_i,l_j,h_g}\}],$$

where, α is a constant. If $\mathcal{X}=\mathcal{R}$; then $\alpha\in\mathcal{R}$; *if* $\mathcal{X}=\{0,1\}$ then $\alpha\in\{0,1\}$; if $\mathcal{X}$ is a set of logical variables, propositions, predicates, or IFPs, then α has sense only when it is some operation "negation". In the latest case, the operation has the form

$$\neg A=[K, L, H, \{\neg a_{k_i,l_j,h_g}\}]$$

4.7 Termwise Subtraction [3]

$$A-(.)\,B=A\underset{(.)}{\oplus}(-1).B=[K\cup P, L\cup Q, H\cup R, \{C_{t_u,v_w,x_y}\}]$$

where,

$$\{C_{t_u,v_w,x_y}\}=\begin{cases} a_{k_i,l_j,h_g}, & \text{if } t_u=k_i\in K, v_w=l_j\in L \text{ and } x_y=h_g\in H-R \\ & \text{or } t_u=k_i\in K, v_w=l_j\in L-Q \text{ and } x_y=h_g\in H \\ & t_u=k_i\in K-p, v_w=l_j\in L \text{ and } x_y=h_g\in H \\ -b_{p_r,q_s,e_d}, \ \text{if } A,B\in 3D-EIM_R & \text{if } t_u=p_r\in P, v_w=q_s\in Q \text{ and } x_y=e_d\in R-H \\ 0 \text{ if } A,B\in 3D-EIM_{\{0,1\}} & \text{if } t_u=p_r\in P, v_w=q_s\in Q-L \text{ and } x_y=e_d\in R \\ \neg b_{p_r,q_s,e_d}, \ \text{if } A,B\in 3D-EIM_P & \text{if } t_u=p_r\in P-K, v_w=q_s\in Q \text{ and } x_y=e_d\in R \\ & \text{if } A,B\in 3D-EIM_{IFP} \\ a_{k_i,l_j,h_g}\circ b_{p_r,q_s,e_d}, & \text{if } t_u=k_i=p_r\in K\cap P, v_w=l_j=q_s\in L\cap Q \\ & \text{and } x_y=h_g=e_d\in H\cap R \\ 0 & \text{for } A,B\in 3D-EIM_R \text{ other wise} \\ & \text{or } A,B\in 3D-EIM_{\{0,1\}} \\ F & \text{for } A,B\in 3D-EIM_P \\ \{0,1\} & \text{for } A,B\in 3D-EIM_{IFP} \end{cases}$$

where,

$$\alpha\circ\beta=\begin{cases} \alpha-\beta \text{ if } A\in 3D-EIM_R \\ \max(0,\alpha-\beta),\ \text{if } A\in 3D-EIM_{\{0,1\}} \\ \alpha\wedge\neg\beta \text{ or } \vee\neg\beta, \text{ if } A\in 3D-EIM_P \text{ or } A,B\in 3D-EIM_{IFP} \end{cases}$$

5. Operation "Reduction" over 3D-EIM

In analogous to [7], the operations "reduction", "projection" and "substitution" over an 3D-EIM, have been introduced. Firstly, we introduce operations $(k,\perp,\perp)$ – $(\perp, l, \perp)$ – and $(\perp,\perp,h)$ – reduction of a given 3D-EIM $A=[K, L, H, \{a_{k_i,l_j,h_g}\}]$:

$$A_{(k,\perp,\perp)}=[K-\{k\}, L, H, \{C_{t_w,v_w,x_y}\}]$$

where

$$C_{t_u,v_w,x_y}=a_{k_i,l_j,h_g} \text{ for } t_u=k_i\in K, v_w=l_j\in L \text{ and } x_y=h_g\in H.$$

$$A_{(\perp,l,\perp)}=[K, L-\{l\}, H, \{C_{t_u,v_w,x_y}\}],$$

where,

$$C_{t_u,v_w,x_y}=a_{k_i,l_j,h_g} \text{ for } t_u=k_i\in K-\{k\}, v_w=l_j\in L \text{ and } x_y=h_g\in H,$$

and $A_{(\perp,\perp,h)}=[K,L,H-\{h\},\{C_{t_u,v_w,x_y}\}]$

where,

$C_{t_u,v_w,x_y}=a_{k_i,l_j,h_g}$ for $t_u=k_i\in K, v_w=l_j\in L$ and $x_y=h_g\in H-\{h\}$

Secondly, define

$$A_{(k,l,h)}=\left(\left(A_{(k,\perp,\perp)}\right)_{(\perp,\perp,h)}\right)_{(\perp,\perp,h)},$$

i.e.,

$$A_{(k,l,h)}=[K-\{k\},L-\{l\},H-\{h\},\{C_{t_u,v_w,x_y}\}]$$

where,

$C_{t_u,v_w,x_y}=a_{k_i,l_j,h_g}$ for $t_u=k_i\in K-\{k\}, v_w=l_j\in L-\{l\}$ and $x_y=h_g\in H-\{h\}$. For every 3D-EIM A and for every k_1, $k_2\in K$, $l_1,l_2\in L$, $h_1,h_2\in H$

$$\left(A_{(k_2,l_2,h_2)}\right)_{(k_2,l_2,h_2)}=\left(A_{(k_2,l_2,h_2)}\right)_{(k_2,l_2,h_2,)}.$$

Thirdly, let $=\{p_1,p_2,\dots p_s\}\subseteq K$, $Q=\{q_1,q_2,\dots q_t\}\subseteq L$ and $R=\{r_1,r_2,\dots r_u\}\subseteq H$, $p\in K$, $l\in L$, $h\in H$, Now, define the following four operations:

$$A_{(p,l,h)}=\left(\cdots\left(\left(A_{(p_1,l,h)}\right)_{(p_2,l,h)}\right)\cdots\right)_{(p_s,l,h)},$$

$$A_{(k,Q,h)}=\left(\cdots\left(\left(A_{(k,l_2,h)}\right)_{(k,l_2,h)}\right)\cdots\right)_{(k,l_t,h)},$$

$$A_{(k,q,H)}=\left(\cdots\left(\left(A_{(k,l,r_2)}\right)_{(k,l,r_2)}\right)\cdots\right)_{(k,l,r_u)},$$

$$A_{(P,Q,H)}=\left(\cdots\left(\left(A_{(p_1,Q,H)}\right)_{(P_2,Q,H)}\right)\cdots\right)_{(p_S,Q,H)}$$

$$=\left(\cdots\left(\left(A_{(p,q_1,H)}\right)_{(p,q_2,H)}\right)\cdots\right)_{(p,q_t,H)}$$

$$=\left(\cdots\left(\left(A_{(p,Q,r_1)}\right)_{(p,Q,r_2)}\right)\cdots\right)_{(P,Q,r_u)}$$

Obviously,

$$A_{(K,L,H)}=I_{\varnothing},$$

$$A_{(\varnothing,\varnothing,\varnothing)}=A.$$

6. Operation "Projection" over an EIM

Let $P\subseteq K$, $Q\subseteq L$, $R\subseteq H$. Then,

$$p^r{}_{P,Q,R}A=[P,Q,R,\{b_{k_i,l_j,h_g}\}],$$

where

$$(\forall k_i\in P)(\forall l_j\in Q)(\forall h_g\in R)\left(b_{k_i,l_j,h_g}=a_{k_i,l_j,h_g}\right)$$

7. Operation "Substitution" over 3D-EIM

Let the 3D-EIM $A=[K,L,H,\{a_{k,l,h}\}]$ be given. First, local substitution over the EIM is defined for the pairs of indices (p,k) and / or (q,l) and / or (r,h) respectively, by

$$\left[\frac{p}{k};\perp;\perp\right]A=\left[\left(K-\{k\}\right)\cup\{p\},L,H,\{a_{k_i,l_j,h_g}\}\right],$$

$$\left[\perp;\frac{q}{l};\perp\right]A=\left[K,\left(L-\{l\}\right)\cup\{q\},H,\{a_{k_i,l_j,h_g}\}\right],$$

$$\left[\perp;\perp;\frac{r}{h}\right]A=\left[K,L,\left(H-\{h\}\right)\cup\{r\},\{a_{k_i,l_j,h_g}\}\right].$$

Second,

$$\left[\frac{p}{k},\frac{q}{l},\frac{r}{h}\right]A=\left[\frac{p}{k};\perp;\perp\right]\left[\perp;\frac{q}{l};\perp\right]\left[\perp;\perp;\frac{r}{h}\right]A$$

$$=\left[\left(K-\{k\}\right)\cup\{p\},\left(L-\{l\}\right)\cup\{q\},\left(H-\{h\}\right)\cup\{r\}\{a_{k_i,l_j,h_g}\}\right]$$

Let the sets of indices $P=\{p_1,p_2,\dots p_m\}$, $Q=\{q_1,q_2,\dots q_n\}$, and $R=\{r_1,r_2,\dots r_s\}$ be given, where $m=card(K)$, $n=card(L)$, $s=card(H)$. Third, for them we define sequentially:

$$\left[\frac{p}{k};\perp;\perp\right]A=\left[\frac{p_1}{k_1}\frac{p_2}{k_2}\cdots\frac{p_m}{k_m};\perp;\perp\right]A,$$

$$\left[\perp;\frac{Q}{L};\perp\right]A=\left[\perp;\frac{q_1}{l_1}\frac{q_2}{l_2}\cdots\frac{q_n}{l_n};\perp;\right]A,$$

$$\left[\perp;\perp;\frac{R}{H}\right]A=\left[\perp;\perp;\frac{r_1}{h_1}\frac{r_2}{h_2}\cdots\frac{r_s}{h_s}\right]A,$$

$$\left[\frac{P}{K};\frac{Q}{L};\frac{R}{H}\right]A=\left[\frac{p_1}{k_1}\frac{p_2}{k_2}\cdots\frac{p_m}{k_m};\frac{q_1}{l_1}\frac{q_2}{l_2}\cdots\frac{q_n}{l_n};\frac{r_1}{h_1}\frac{r_2}{h_2}\cdots\frac{r_s}{h_s}\right]$$

$$A=[P,Q,R,\{a_{k,l,h}\}]$$

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